On the Integration of Differential Equations *

Leonhard Euler

§1 Here, I consider differential equations of first degree, which involve only two variables and which therefore can be represented in this general form

$$Mdx + Ndy = 0,$$

if M and N denote any arbitrary functions of the two variables x and y. But it was proved that an equation of this kind always expresses a certain relation among the variables x and y, by which for each value of the one the values for the other are defined. But because this finite relation among the variables must be found by means of integration, the complete integral equation will receive a constant quantity, which, since it is completely arbitrary, contains infinitely many particular integral equations, which all likewise satisfy the differential equation.

§2 Therefore, having propounded any differential equation of this kind

$$Mdx + Ndy = 0$$

the whole task of the Analysis consists is to find a finite equation between the variables x and y, which equation expresses the same relation as the differential equation, and this in the broadest sense, such that it contains a

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certain arbitrary constant, which is not found in the differential equation. But if this question is propounded in this most general form, until now no way was found to answer it; and all cases, which could be solved until now, can be reduced to a tiny number, such that in this part of Analysis, as in the remaining ones, still very huge progress is desired; and therefore, the complete cognition of all secrets of this science can never be expected.

§3 Almost everything achieved in this task until now can be reduced to these cases, in which the differential equation

$$Mdx + Ndy = 0$$

either immediately admits a separation of variables, or by means of suitable substitutions can be reduced to such a form. For, if by introducing two new variables v and z instead of x and y the propounded differential equation can be transformed into a form of this kind

$$Vdv + Zdz = 0,$$

in which V is a function only of v and Z only of z, the whole problem will be solved, since the complete integral equation will be:

$$\int Vdv + \int Zdz = \text{Const.},$$

which obviously contains that arbitrary constant, since it will be introduced by the integration. And almost all artifices, which the Analysts have used in the resolution of equations of this kind, reduce to this.

§4 Therefore, if the propounded differential equation does not immediately admit the separation of variables, the whole task used to be that appropriate substitutions, which lead to a separable equation, are investigated, where often the highest ingeniousness, which the Geometers showed to reach his goal, must be admired. Since there is nevertheless no certain way known to find substitutions of this kind, this method seems rather unnatural, whence I decided, to consider a maybe not new, but nevertheless still not sufficiently developed method in more detail; since this method does not require substitutions, it seems to be more natural; furthermore, it is based on the nature of differentials and even contains the first method, at least partially, as a special case.

§5 Having reduced the differential equation to this form

$$Mdx + Ndy = 0$$

consider the formula $Mdx + Bdy$ without taking into account that is has to vanish, and examine, whether it is a differential of a certain function of x and y or not. It is to be examined in such a way, as it was explained already abundantly in many different papers; of course, both functions M and N must be differentiated, and since their differentials must have a form of this kind

$$dM = pdx + qdy \quad \text{and} \quad dN = rdx + sdy,$$

see, whether it is $q = r$ or not. For, if it was $q = r$, this is an infallible criterion that the formula $Mdx + Ndy$ is integrable: But if it was not $q = r$, it is equally certain that this formula did not result from a differentiation of a certain finite function of x and y. Therefore, the whole question is reduced to two cases, the one, in which it is $q = r$, the other, in which these two quantities q and r are not equal to each other.

§6 Therefore, to see the equality, or inequality, of the quantities q and r, it is not even necessary that the functions M and N are expanded completely by differentiation, but it suffices to consider the quantity x as constant in the function M, which is connected to dx, and only ask for its partial differential, which is obtained from the variability of y only, since this way the term qdy is obtained, but I usually denote the value of q found this way by the sign $\left(\frac{dM}{dy}\right)$. In like manner, differentiate the other function N, which is connected to dy, in such a way, that y is treated as a constant, and from the variability of x only the part rdx is obtained, where I equally express the value of r by $\left(\frac{dN}{dx}\right)$. Therefore, if the formula $Mdx + Ndy$ was of such a nature that it is

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

it is integrable, and its integral can be found the way we will explain in the following problem. Having done this, let us see, how one has to proceed, if the condition of criterion is not satisfied,

Problem 1

§7 If the differential equation

$$Mdx + Ndy = 0$$

was of such a nature that it is

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

to find its integral equation.

SOLUTION

If it was

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

then a finite function of the two variables x and y is given, which, having differentiated it, yields $Mdx + Ndy$. Let V be this function, and because it is

$$dV = Mdx + Ndy,$$

Mdx will be the differential of V, if only x is taken as a variable, and Ndy its differential, if only y is taken as a variable. Therefore, hence vice versa V will be found, if either Mdx is integrated, having considered y as constant, or Ndy is integrated, having considered x as constant: And so this operation is reduced to the integration of a differential formula involving only one variable which is postulated to be possible in this case, no matter whether it succeeds algebraically or requires the quadrature of curves. But because this way the quantity V is found in two ways, and the one integration leads to an arbitrary function of y instead of the constant, the other on the other hand yields a function of x instead of the constant, such that it is

$$\text{both} \quad V = \int Mdx + Y \quad \text{and} \quad V = \int Ndy + X,$$

it is always possible to define these functions Y of y and X of x in such a way that it is $\int Mdx + Y = \int Ndy + X$, which is easily achieved in each case. Since, having done this, the quantity V is the integral of the formula $Mdx + Ndy$, it is evident that the integral equation of the propounded equation $Mdx + Ndy = 0$ will be $V =$ Const., and it will be the complete integral equation, since it involves an arbitrary constant.

Corollary 1

§8 In this case the case of separable equations is immediately contained. For, if M was a function of x only, and N a function of y only, it will certainly be

$$\left(\frac{dM}{dy}\right)=0 \quad \text{and} \quad \left(\frac{dN}{dx}\right)=0 \quad \text{and hence} \quad \left(\frac{dM}{dy}\right)=\left(\frac{dN}{dx}\right);$$

therefore, this is the simplest case the problem contains.

Corollary 2

§9 But if in the differential equation

$$Mdx+Ndy=0$$

M was a function of x only, and N a function only of y, each of both parts is integrable separately, and the integral equation will be:

$$\int Mdx+\int Ndy=\text{Const.}$$

Corollary 3

§10 Furthermore, our problem provides us with the solution of infinitely many other differential equations, the common character of all of which is that it is

$$\left(\frac{dM}{dy}\right)=\left(\frac{dN}{dx}\right),$$

and their resolution is possible by means of integration of formulas containing one single variable.

Scholium 1

§11 Therefore, if in the differential equation $Mdx+Ndy=0$ it was $\left(\frac{dM}{dy}\right)=\left(\frac{dN}{dx}\right)$, its resolution has no difficulty, as long as the integration of formulas involving one variable is assumed to be possible; this can certainly justly be postulated. Nevertheless, the determination of those functions X and Y which have to be introduced instead of the constants, could seem to create certain inconveniences; but this inconvenience will soon be found to vanish in the single cases. But to contract this operation even further, the second integration is not even necessary. For, after the one part Mdx, having considered y as constant, was integrated, which integral we want to put $= Q$, set

$$V = Q + Y,$$

having meanwhile put Y for a indefinite function of y, which the other variable x does not enter at all. Then differentiate this quantity $Q + Y$ again treating x as a constant, and since the differential has to result $= Ndy$, from this condition the function Y will be determined most easily, since from the nature of the procedure itself the quantity x will be eliminated immediately. But having found this function Y, the integral equation will be $Q + Y =$ Const., which operation will be conveniently illustrated by the following examples.

EXAMPLE 1

§12 *To integrate this differential equation:*

$$2axydx + axxdy - y^3dx - 3xyydy = 0.$$

Having compared this equation to the form $Mdx + Ndy = 0$, it will be:

$$M = 2axy - y^3 \quad \text{and} \quad N = axx - 3xyy.$$

Therefore, first it is to be checked, whether this case is contained in the problem, for which purpose we need the values:

$$\left(\frac{dM}{dy}\right) = 2ax - 3yy \quad \text{and} \quad \left(\frac{dN}{dx}\right) = 2ax - 3yy;$$

because they are equal, the prescribed operation will necessarily succeed. But, haven taken y to be constant, one will find:

$$\int Mdx = axxy - y^3x + Y;$$

if the differential of this form is taken, having put x to be constant, this equation will result:

$$axxdy - 3yyxdy + dY = Ndy,$$

and having substituted its value $axx - 3xyy$ for N again, it will be $dY = 0$, whence $Y = 0$ results, or $Y =$ Const.. Hence one will have the integral equation is question; it is:

$$axxy - y^3x = \text{Const.}.$$

EXAMPLE 2

§13 *To integrate this differential equation:*

$$\frac{ydy + xdx - 2ydx}{(y-x)^2} = 0.$$

Having compared this equation to the form $Mdx + Ndy = 0$, it will be:

$$M = \frac{x-2y}{(y-x)^2} \quad \text{and} \quad N = \frac{y}{(y-x)^2}.$$

Now, that it becomes plain, whether this equation is contained in the case of the problem, find the differentials:

$$\left(\frac{dM}{dy}\right) = \frac{2y}{(y-x)^2} \quad \text{and} \quad \left(\frac{dN}{dx}\right) = \frac{2y}{(y-x)^2};$$

since they are equal, the task will be successful. Hence having taken y to be constant according to the rule one concludes the integral to be:

$$\int Mdx = \int \frac{xdx - 2ydx}{(y-x)^2} = -\int \frac{dx}{y-x} - \int \frac{ydx}{(y-x)^2}$$

and one will find:

$$\int Mdx = \log(y-x) - \frac{y}{y-x} + Y,$$

whose differential, having taken x to be constant, must produce the other part Ndy of the propounded equation; hence one will have:

$$Ndy = \frac{dy}{y-x} + \frac{xdy}{(y-x)^2} + dY = \frac{ydy}{(y-x)^2} + dY.$$

Therefore, because it is

$$Ndy = \frac{ydy}{(y-x)^2}, \quad \text{it will} \quad dY = 0 \quad \text{and} \quad Y = 0,$$

for, the constant Y can be neglected, since it has already been introduced in the integral equation, which will be:

$$\log(y-x) - \frac{y}{y-x} = \text{Const.}$$

EXAMPLE 3

§14 *To integrate this differential equation:*

$$\frac{dx}{x} + \frac{yydx}{x^3} - \frac{ydy}{xx} + \frac{(ydx - xdy)\sqrt{xx+yy}}{x^3} = 0.$$

Having compared this equation to the form $Mdx + Ndy = 0$, we will have:

$$M = \frac{xx + yy + y\sqrt{xx+yy}}{x^3} \quad \text{and} \quad N = \frac{-y - \sqrt{xx+yy}}{xx},$$

whence checking the conditions of the criterion we find:

$$\left(\frac{dM}{dy}\right) = \frac{2y}{x^3} + \frac{\sqrt{xx+yy}}{x^3} + \frac{yy}{x^3\sqrt{xx+yy}}$$

and

$$\left(\frac{dN}{dx}\right) = \frac{2y}{x^3} + \frac{2\sqrt{xx+yy}}{x^3} - \frac{x}{xx\sqrt{xx+yy}},$$

since which values having simplified them become equal to each other, of course

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right) = \frac{2y}{x^3} + \frac{xx+2yy}{x^3\sqrt{xx+yy}},$$

the resolution will be possible. Therefore, having assumed y to be constant, investigate:

$$\int Mdx = \log x - \frac{yy}{2xx} + y\int \frac{dx}{x^3}\sqrt{xx+yy}.$$

But by means of the rules how to integrate formulas involving one variable, since here y is considered to be constant, one finds:

$$\int \frac{ydx}{x^3}\sqrt{xx+yy} = \frac{-y\sqrt{xx+yy}}{2xx} + \frac{1}{2}\log\frac{\sqrt{xx+yy}-y}{y},$$

such that it is:

$$\int Mdx = \log x - \frac{yy}{2xx} - \frac{y\sqrt{xx+yy}}{2xx} + \frac{1}{2}\log\frac{\sqrt{xx+yy}-y}{y} + Y.$$

But since the differential of this quantity, having assumed x to be constant, has to yield

$$Ndy = -\frac{-ydy - dy\sqrt{xx+yy}}{xx},$$

we will obtain:

$$Ndy = \frac{-ydy}{xx} - \frac{dy\sqrt{xx+yy}}{2xx} - \frac{yydy}{2xx\sqrt{xx+yy}} - \frac{dy}{2y} - \frac{dy}{2\sqrt{xx+yy}} + dY,$$

having compared this form to that one it will be:

$$dY = -\frac{dy\sqrt{xx+yy}}{2xx} + \frac{yydy}{2xx\sqrt{xx+yy}} + \frac{dy}{2y} + \frac{dy}{2\sqrt{xx+yy}},$$

where the terms, which still contain x, cancel each other, such that it is

$$dY = \frac{dy}{2y} \quad \text{and} \quad Y = \frac{1}{2}\log y.$$

Having found this value for Y, one will obtain the integral equation in question:

$$\log x - \frac{yy}{2xx} - \frac{y\sqrt{xx+yy}}{2xx} + \frac{1}{2}\log(\sqrt{xx+yy}-y) = \text{Const.}$$

Scholium 2

§15 From these examples it is already understood, how the prescribed operation is to be done, such that hence nothing causes any further inconveniences, except for those always remaining from the integration of formulas involving one variable, when the integration cannot be done algebraically and does not admit it to be reduced to the quadrature of the circle or hyperbola. But then the above quadratures have to be treated in like manner, and if these difficulties remain, they are not to be ascribed to this method. Therefore, it is possible to assume here, if the differential equation

$$Mdx + Ndy = 0$$

was of such a nature that in it it is

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

that its integration is possible for us; therefore, I proceed to equations, in which this criterion is not satisfied.

THEOREM

§16 If in the differential equation

$$Mdx + Ndy = 0$$

it was not

$$\left(\frac{dM}{dx}\right) = \left(\frac{dN}{dx}\right),$$

always a multiplicator is given, multiplied by which the formula $Mdx + Ndy$ becomes integrable.

PROOF

Since it is not

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

the formula $Mdx + Ndy$ will also not be integrable, or no function of x and y exists, whose differential is $Mdx + Ndy$. But here not the integral so of the formula $Mdx + Ndy$ as of the equation $Mdx + Ndy = 0$ is in question; and because the same equation holds, if it is multiplied by any function L of x and y, such that it is

$$LMdx + LNdy = 0,$$

it is to be demonstrated that always a function L of such a kind is given that the formula

$$LMdx + LNdy$$

becomes integrable. For this to happen it is necessary that it is:

$$\left(\frac{d.LM}{dy}\right) = \left(\frac{d.LN}{dx}\right),$$

or if one puts

$$dL = Pdx + Qdy,$$

because it is

$$\left(\frac{dL}{dy}\right) = Q \quad \text{and} \quad \left(\frac{dL}{dx}\right) = P,$$

the function L has to be of such a nature that it is:

$$L\left(\frac{dM}{dy}\right) + MQ = L\left(\frac{dN}{dx}\right) + NP.$$

But it is evident that this condition suffices to define the function L in such a way that, if the formula $Mdx + Ndy$ is multiplied by it, becomes integrable.

COROLLARY 1

§17 Therefore, having found the multiplicator L rendering the formula

$$Mdx + Ndy$$

integrable, the equation $Mdx + Ndy = 0$, having reduced it to the form

$$LMdx + LNdy = 0,$$

can be integrated by the method explained in the preceding problem.

COROLLARY 2

§18 Having considered y as constant let the integral $\int LMdx$ be in question, and add such a function Y of y to it that, if the aggregate

$$\int LMdx + Y$$

is differentiated again, having considered x as constant now, $LNdy$ results. Having done this the integral equation will be

$$\int LMdx + Y = \text{Const.}$$

Corollary 3

§19 Therefore, the multiplicator must be of such a nature that having put

$$dL = Pdx + Qdy$$

this equation is satisfied:

$$L\left(\frac{dM}{dy}\right) + MQ = L\left(\frac{dN}{dx}\right) + NP$$

or this one:

$$\frac{NP - MQ}{L} = \left(\frac{dM}{dy}\right) - \left(\frac{dN}{dx}\right),$$

whence it is obvious, if it would be

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right),$$

that one can take 1 for L, or any constant quantity, while it is $P = 0$ and $Q = 0$.

Scholium

§20 Therefore, if hence the multiplicator L could be found in general, one would have the general resolution of all differential equation of first order, what can certainly not be hoped for. Therefore, we have to be content, if we are able to find multiplicators of this kind for the various cases and many classes of differential equations. But there are two classes of equations, for which such multiplicators can be found conveniently, of which the one comprehends the equations, in which the one variable never has more than one dimension; the other class on the other hand is the class of homogeneous equations. But except for these two classes there are many other cases, in which the invention of such a factor is possible; and it will not be useless to have examined all of them, since this seems to open the only way to develop and expand this branch of Analysis, which is still desired. Therefore, I decided to collect many classes of equations, which can be rendered integrable by means of a multiplicator of this kind.

PROBLEM 2

§21 Having known one single multiplicator L rendering the formula $Mdx + Ndy$ integrable, to find infinitely many other multiplicators, which do the same.

SOLUTION

Because the formula $L(Mdx + Ndy)$ is integrable by assumption, let its integral be $= z$, such that it is

$$dz = L(Mdx + Ndy),$$

where z is a certain function of x and y. Now, let Z denote any function of z, and since the formula Zdz is also integrable, because of

$$Zdz = LZ(Mdx + Ndy)$$

it is obvious that the propounded formula $Mdx + Ndy$ also becomes integrable, if it is multiplied by LZ. Therefore, having found one multiplicator L rendering the formula $Mdx + Ndy$ integrable, from it innumerable other factors LZ can be found, which will enable the same by taking any arbitrary function of the integral

$$\int L(Mdx + Ndy)$$

for Z.

COROLLARY 1

§22 Therefore, having propounded any differential formula $Mdx + Ndy$, not only one but even infinitely many multiplicators are given rendering it integrable. But it suffices to have found one of them, since all remaining ones are determined by this one.

COROLLARY 2

§23 Therefore, if one has the differential equation

$$Mdx + Ndy = 0,$$

it can be rendered integrable in infinitely many ways. But no matter whether one takes the multiplicator L, or any other one LZ, the found integral equation reduces to the same; for, since that factor L yields $z = \text{Const.}$, but this on the other hand $\int Zdz = \text{Const.}$, they are identical, since $\int Zdz$ is a function of z.

EXAMPLE 1

§24 *To find all multiplicators rendering this formula*

$$\alpha ydx + \beta xdy$$

integrable.

One multiplicator solving this problem is obvious, namely $\frac{1}{xy}$. Therefore, let $L = \frac{1}{xy}$, and further

$$dz = \frac{\alpha ydx + \beta xdy}{xy} = \frac{\alpha dx}{x} + \frac{\beta dy}{y},$$

whence by integrating this equation results

$$z = \alpha \log x + \beta \log y = \log x^{\alpha} y^{\beta}.$$

Now, let Z denote any function of $z = \log x^{\alpha} y^{\beta}$, this means of $x^{\alpha} y^{\beta}$, and all multiplicators in question will be contained in this general form

$$\frac{1}{xy}\text{funct. } x^{\alpha} y^{\beta}.$$

Therefore, one will find simpler multiplicators, if the arbitrary $x^{\alpha} y^{\beta}$ is taken instead of the function; and so the formula $\alpha ydx + \beta xdy$ is rendered integrable by this further extending multiplicator $x^{\alpha n-1} y^{\beta n-1}$. If more composite ones are desired, one will be able to combine any number of formulas of this kind that one has

$$Ax^{\alpha n-1} y^{\beta n-1} + Bx^{\alpha m-1} y^{\beta m-1} + \text{etc.}$$

EXAMPLE 2

§25 *To find all multiplicators rendering this differential formula*

$$\alpha x^{\mu-1}y^{\nu}dx + \beta x^{\mu}y^{\nu-1}dy$$

integrable.

Here, again one multiplicator immediately reveals itself to us

$$L = \frac{1}{x^{\mu}y^{\nu}},$$

which yields

$$dz = \frac{\alpha dx}{x} + \frac{\beta dy}{y},$$

whence it is

$$z = \alpha \log x + \beta \log y = \log x^{\alpha}y^{\beta}.$$

Therefore, having put Z for any function of $x^{\alpha}y^{\beta}$, all multiplicators will be contained in this general expression

$$\frac{Z}{x^{\mu}y^{\nu}} = \frac{1}{x^{\mu}y^{\nu}}\text{funct. } x^{\alpha}y^{\beta}.$$

If instead of this function one takes an arbitrary power $x^{\alpha n}y^{\beta n}$, one will hence obtain innumerable multiplicators, consisting of one single term $x^{\alpha n-\mu}y^{\beta n-\nu}$, by taking any arbitrary numbers for n.

SCHOLIUM

§26 Therefore, it can happen that two or more differential formulas of this kind

$$\alpha x^{\mu-1}y^{\nu}dx + \beta x^{\mu}y^{\nu-1}dy$$

have the same multiplicator: If this happens, the differential equation composed of formulas of this kind as terms can be rendered integrable, if this common multiplicator is applied. Let us expand this case already considered once.

PROBLEM 3

§27 Let this differential equation be propounded:

$$\alpha y dx + \beta x dy + \gamma x^{\mu-1} y^{\nu} dx + \delta x^{\mu} y^{\nu-1} = 0,$$

whose integral is to be found.

SOLUTION

To find an appropriate multiplicator such that this equation is rendered integrable, consider both terms separately. And we certainly saw the first term $\alpha y dx + \beta x dy$ to be rendered integrable by this multiplicator

$$x^{\alpha n-1} y^{\beta n-1},$$

but the second term $\gamma x^{\mu-1} y^{\nu} dx + \delta x^{\mu} y^{\nu-1} dy$ by this one

$$x^{\gamma m-\mu} y^{\delta m-\nu}.$$

Since now for n and m any arbitrary numbers can be taken, these two factors can be made equal; hence it is

$$\alpha n - 1 = \gamma m - \mu \quad \text{and} \quad \beta n - 1 = \delta m - \nu$$

and hence

$$n = \frac{\gamma m - \mu + 1}{\alpha} = \frac{\delta m - \nu + 1}{\beta},$$

and hence one obtains

$$m = \frac{\alpha\nu - \beta\mu - \alpha + \beta}{\alpha\delta - \beta\gamma} \quad \text{and} \quad n = \frac{\gamma\nu - \delta\mu - \gamma + \delta}{\alpha\delta - \beta\gamma}.$$

Having found these values for m and n, this common multiplicator will give this integral equation:

$$\frac{1}{n} x^{\alpha n} y^{\beta n} + \frac{1}{m} x^{\gamma m} y^{\delta m} = \text{Const.}$$

COROLLARY 1

§28 Therefore, this integral equation is always algebraic, if the true values are found for m and n. Therefore, only those cases need a particular reduction, in which the number m and n become infinite or vanish.

COROLLARY 2

§29 But the two numbers m and n become infinite, if it was $\alpha\delta = \beta\gamma$. But in this case the differential is resolved into two factors, and acquires this form

$$(\alpha y dx + \beta x dy)(1 + \frac{\gamma}{\alpha} x^{\mu-1} y^{\nu-1}) = 0$$

and hence it will be

$$\text{either} \quad \alpha y dx + \beta x dy = 0, \quad \text{or} \quad 1 + \frac{\gamma}{\alpha} x^{\mu-1} y^{\nu-1} = 0,$$

none of which two resolutions is of any difficulty.

COROLLARY 3

§30 But if it is $n = 0$ or

$$\gamma(\nu - 1) = \delta(\mu - 1),$$

consider the number n as very small, and because by means of a convergent series it is

$$x^{\alpha n} = 1 + \alpha n \log x + \frac{1}{2}\alpha^2 n^2 (\log x)^2 + \text{etc.} \quad \text{and} \quad y^{\beta n} = 1 + \beta n \log y + \frac{1}{2}\beta^2 n^2 (\log y)^2 + \text{etc.},$$

it will be

$$\frac{1}{n} x^{\alpha n} y^{\beta n} = \frac{1}{n} + \alpha \log x + \beta \log y = \log x^{\alpha} y^{\beta}$$

having included the first part $\frac{1}{n}$ into the constant. Therefore, in this case the integral equation will be:

$$\log x^{\alpha} y^{\beta} + \frac{1}{m} x^{\gamma m} y^{\delta m} = \text{Const.}$$

COROLLARY 4

§31 Therefore, for this case set

$$\mu = \gamma k + 1 \quad \text{and} \quad \nu = \delta k + 1,$$

that one has this differential equation:

$$\alpha y dx + \beta x dy + \gamma x^{\gamma k} \delta^{\delta k+1} dx + \delta x^{\gamma k+1} y^{\delta k} dy = 0,$$

and because it is

$$m = \frac{\alpha\delta k - \beta\gamma k}{\alpha\delta - \beta\gamma} = k,$$

the integral equation will be

$$\log x^{\alpha} y^{\beta} + \frac{1}{k} x^{\gamma k} y^{\delta k} = \text{Const.}$$

COROLLARY 5

§32 In like manner, if it was $m = 0$ or

$$\alpha(\nu - 1) = \beta(\mu - 1),$$

because of

$$\frac{1}{m} x^{\gamma m} y^{\delta m} = \log x^{\gamma} y^{\delta},$$

if one puts $\mu = \alpha k + 1$ and $\nu = \beta k + 1$, whence it is

$$n = \frac{\gamma\beta k - \delta\alpha k}{\alpha\delta - \beta\gamma} = -k,$$

the integral of this equation

$$\alpha y dx + \beta x dy + \gamma x^{\alpha k} y^{\beta k+1} dx + \delta x^{\alpha k+1} y^{\beta k} dy = 0$$

will be

$$-\frac{1}{k} x^{-\alpha k} y^{-\beta k} + \log x^{\gamma} y^{\delta} = \text{Const.}$$

Scholium

§33 But a resolution of this kind into several terms, which are rendered integrable by means of the same multiplicator, does not extend to equations of all classes. For, it can certainly happen that the whole equation, if it was multiplied by a certain quantity, becomes integrable, although no single term of it is integrable separately, from which one must not attribute to much to these considerations.

Problem 4

§34 If this differential equation is propounded

$$Pdx + Qydx + Rdy = 0,$$

where P, Q and R denote any arbitrary functions of x, such that the other variable y does not have more than one dimension, to find the multiplicator, which renders it integrable.

Solution

Having compared this equation to the form $Mdx + Ndy = 0$ it will be

$$M = P + Qy \quad \text{and} \quad N = R,$$

whence it will be

$$\left(\frac{dM}{dy}\right) = Q \quad \text{and} \quad \left(\frac{dN}{dx}\right) = \frac{dR}{dx}.$$

Now set L for the multiplicator in question, and let

$$dL = pdx + qdy,$$

and this equation must be satisfied

$$\frac{Np - Mq}{L} = Q - \frac{dR}{dx} = \frac{Rp - (P + Qy)q}{L}.$$

Because now $Q - \frac{dR}{dx}$ is a function of x only, one will also be able to take a function of x only for L, such that it is $q = 0$, and $dL = pdx$; hence it will be:

$$Q - \frac{dR}{dx} = \frac{Rp}{L} \quad \text{or} \quad Qdx - dR = \frac{RdL}{L}$$

and hence

$$\frac{dL}{L} = \frac{Qdx}{R} - \frac{dR}{R}.$$

Hence by integrating one will have

$$\log L = \int \frac{Qdx}{R} - \log R,$$

and having taken e for the number whose hyperbolic logarithm is 1 this expression results

$$L = \frac{1}{R} e^{\int \frac{Qdx}{R}}.$$

But having found this multiplicator the integral equation will be:

$$\int \frac{Pdx}{R} e^{\int \frac{Qdx}{R}} + y e^{\int \frac{Qdx}{R}} = \text{Const.}$$

COROLLARY 1

§35 If the equation has the propounded form, it, before it is treated this way, can be divided by R that it obtains this form

$$Pdx + Qydx + dy = 0,$$

or one can immediately assume $R = 1$; having done this the multiplicator will be $e^{\int Qdx}$, and the integral equation

$$\int e^{\int Qdx} Pdx + e^{\int Qdx} y = \text{Const.}$$

COROLLARY 2

§36 If one puts this integral

$$\int e^{\int Qdx} Pdx + e^{\int Qdx} y = z,$$

such that z is a certain function of two variables, but then Z denotes any arbitrary function of z, all multiplicators, which render the formula

$$Pdx + Qdy + dy$$

integrable, are contained in this general form $e^{\int Qdx}$.

PROBLEM 5

§37 If this differential equation is propounded:

$$Py^n dx + Qdydx + Rdy = 0,$$

where P, Q and R denote any functions of x, to find a multiplicator rendering it integrable.

SOLUTION

Therefore, it will be $M = Py^n + Qy$ and $N = R$, and hence

$$\left(\frac{dM}{dy}\right) = nPy^{n-1} + Q \quad \text{and} \quad \left(\frac{dN}{dx}\right) = \frac{dR}{dx}.$$

Hence having put the multiplicator in question L and

$$dL = pdx + qdy,$$

from the results found before it will be:

$$\frac{Rp - Py^n q - Qyq}{L} = nPy^{n-1} + Q - \frac{dR}{dx}.$$

Assume $L = Sy^m$, while S is a function of x only, then it will be

$$p = \frac{y^m dS}{dx} \quad \text{and} \quad q = mSy^{m-1};$$

having substituted these values, this equation will result:

$$\frac{RdS}{Sdx} - mPy^{n-1} - mQ = nPy^{n-1} + Q - \frac{dR}{dx}.$$

For this equation to hold, one has to take $m = -n$, and it will be

$$\frac{RdS}{Sdx} = (1-n)Q - \frac{dR}{dx} \quad \text{or} \quad \frac{dS}{S} = \frac{(1-n)Qdx}{R} - \frac{dR}{R}.$$

Hence, because by integration this expression results

$$S = \frac{1}{R} e^{(1-n)\int \frac{Qdx}{R}},$$

because of $m = -n$ the multiplicator in question will be:

$$L = \frac{y^{-n}}{R} e^{(1-n)\int \frac{Qdx}{R}}$$

and the integral equation will be

$$\frac{y^{1-n}}{1-n} e^{(1-n)\int \frac{Qdx}{R}} + \int \frac{Pdx}{R} e^{(1-n)\int \frac{Qdx}{R}} = \text{Const.}$$

COROLLARY 1

§38 If it is $n = 0$, we have the case treated before of the equation

$$Pdx + Qdx + Qydx + Rdy = 0,$$

which by means of the multiplicator

$$\frac{1}{R} e^{\int \frac{Qdx}{R}}$$

is rendered integrable; and the integral equation of it is

$$ye^{\int \frac{Qdx}{R}} + \int \frac{Pdx}{R} e^{\int \frac{Qdx}{R}} = \text{Const.}$$

COROLLARY 2

§39 But let $n = 1$, that the differential equation is:

$$Pydx + Qdydx + Rdy = 0$$

the multiplicator because of $1 - n = 0$ will be $\frac{1}{Ry}$, so that the equation is reduced to this form

$$\int \frac{(P+Q)dx}{R} + \log y = \text{Const.}$$

SCHOLIUM

§40 Additionally, the solution for this problem is easily deduced from the preceding problem. For, divide the propounded differential equation by y^n, and one will have:

$$Pdx + Qy^{1-n}dx + Ry^{-n}dy = 0.$$

Put $y^{1-n} = z$, it will be $(1-n)y^{-n}dy = dz$, and so the equation becomes this one:

$$Pdx + Qzdx + \frac{1}{1-n}Rdz = 0,$$

which is identical to the equation of the preceding problem. Therefore, because these two equations both belong to the case, in which the one variable never has more than one dimension, we have covered it completely by this method of multiplicators. Therefore, I proceed to another class, the class of homogeneous differential equations, which are known that they can also be treated by this method. But for this purpose it is necessary to give a lemma describing the nature of homogeneous functions in advance, if we want to derive the operation on first principles.

LEMMA

§41 If V was a homogeneous function, in which the two variables x and y add up to n dimensions everywhere, its differential

$$dV = Pdx + Qdy$$

will be of such a nature that it is

$$Px + Qy = nV.$$

PROOF

Put $y = xz$, and the function V will obtain a form of this kind $x^n Z$, where Z is a certain function of z only. Therefore, it will hence be

$$dV = nx^{n-1}Zdx + x^n dZ.$$

Reduce these two variables x and z also to the propounded differential $dV = Pdx + Qdy$, and because it is

$$dy = zdx + xdz,$$

it will be

$$dV = (P + Qz)dx + Qxdz;$$

therefore, it is necessary that it is

$$nx^{n-1}Z = P + Qz,$$

and by multiplying by x on both sides:

$$nx^nZ = nV = Px + Qxz = Px + Qy,$$

such that it is $Px + Qy = nV$.

COROLLARY 1

§42 Therefore, since we have the two equations:

$$dV = Pdx + Qdy \quad \text{and} \quad nV = Px + Qy,$$

hence the two functions P and Q can be defined; for, one will find:

$$P = \frac{ydV - nVdy}{ydx - xdy} \quad \text{and} \quad Q = \frac{nVdx - xdV}{ydx - xdy}.$$

COROLLARY 2

§43 Therefore, if V is a homogeneous function of n dimensions, because of

$$P = \left(\frac{dV}{dx}\right) \quad \text{and} \quad Q = \left(\frac{dV}{dy}\right)$$

it will be

$$\left(\frac{dV}{dx}\right) = \frac{ydV - nVdy}{ydx - xdy} \quad \text{and} \quad \left(\frac{dV}{dy}\right) = \frac{nVdx - xdV}{ydx - xdy},$$

where it is to be noted that in these fractions the differentials cancel each other, or both numerators will be divisible by $ydx - xdy$.

Problem 6

§44 Having propounded the differential equation

$$Mdx + Ndy = 0,$$

in which M and N are homogeneous functions of x and y, both of the same number of dimensions, to find the multiplicator rendering the equation integrable.

Solution

Let n be the number of dimensions corresponding to both functions M and N and by means of the preceding paragraph it will be

$$\left(\frac{dM}{dy}\right) = \frac{nMdx - xdM}{ydx - xdy} \quad \text{and} \quad \left(\frac{dN}{dx}\right) = \frac{ydN - nNdy}{ydx - xdy}$$

and hence

$$\left(\frac{dM}{dy}\right) - \left(\frac{dN}{dx}\right) = \frac{n(Mdx + Ndy) - xdM - ydN}{ydx - xdy}.$$

Now, it is easily concluded that a multiplicator is given which is also a homogeneous function of x and y. Therefore, let L be such a homogeneous function of m dimensions. Hence, if in § 19 one puts

$$dL = Pdx + Qdy,$$

it will be [§ 42]

$$P = \frac{ydL - mLdy}{ydx - xdy} \quad \text{and} \quad Q = \frac{mLdx - xdL}{ydx - xdy}$$

and hence, because according to § 19 it must be

$$\frac{NP - MQ}{L} = \left(\frac{dM}{dy}\right) - \left(\frac{dN}{dx}\right),$$

by multiplying by $ydx - xdy$ on both sides one will obtain:

$$\frac{NydL - mLNdy - mLMdx + MxdL}{L} = n(Mdx + Ndy) - xdM - ydN,$$

whence one finds:

$$\frac{dL}{L} = \frac{(m+n)(Mdx+Ndy) - xdM - ydN}{Mx+Ny},$$

which formula obviously becomes integrable having put $m+n=-1$; having done this it will be

$$\log L = -\log(Mx+Ny).$$

Therefore, one will find the multiplicator in question to be

$$L = \frac{1}{Mx+Ny}.$$

COROLLARY 1

§45 Therefore, having propounded the homogeneous differential equation $Mdx+Ndy=0$, it will most easily be rendered integrable, since the formula

$$\frac{Mdx+Ndy}{Mx+Ny}$$

is integrable, whose integral, by means of the method given above, will give the integral equation is question.

COROLLARY 2

§46 An inconvenience occurs only in the case, where it is $Mx+Ny=0$, as it happens in the equation $ydx-xdy=0$, which would have to be divided by

$$xy - xy = 0 \cdot xy.$$

But since any multiple of this divisor equally satisfies, the divisor xy will also solve the task, as it is perspicuous per se.

SCHOLIUM

§47 There is a very well known method, by which the most ingenious Joh. Bernoulli once taught to render all homogeneous differential equations separable. Having propounded an equation of this kind

$$Mdx + Ndy = 0$$

in which M and N are homogeneous functions of n dimensions, he tells us to put $y = ux$; having done this the functions M and N will obtain forms of this kind that it is

$$M = x^n U \quad \text{and} \quad N = x^n V,$$

while U and V are functions of u only. Therefore, the propounded equation, divided by x^n will go over into this one:

$$Udx + Vdy = 0.$$

But because it is $dy = udx + xdu$, we will have

$$Udx + Vudx + Vxdu = 0,$$

which divided by $x(U + Vu)$ becomes separable, or this form

$$\frac{(U + Vu)dx + Vxdu}{x(U + Vu)}$$

becomes integrable. But it is

$$(U + Vu)dx + Vxdu = \frac{1}{x^n}(Mdx + Ndy)$$

and

$$x^n(U + Vu) = M + Nu.$$

Therefore, this formula will be integrable:

$$\frac{Mdx + Ndy}{x(M + Nu)} = \frac{Mdx + Ndy}{Mx + Ny} \quad \text{because of} \quad ux = y.$$

Therefore, having explained these two classes of equations, which can be rendered integrable by means of suitable multiplicators, let us see, to which other classes the same method can be extended: And at first I observe that all differential equations which can be integrated by other methods can also be treated by this method by means of a suitable multiplicator, which will be explained more clearly in the following problem.

Problem 7

§48 Having propounded the differential equation $Mdx + Ndy = 0$, if its complete integral equation was found, to assign all multiplicators rendering the differential equation integrable.

Solution

Since the complete integral equation involves an arbitrary constant C, which does not occur in the differential equation, no matter how intricate it might be, find its value by resolution of the equation, which we want to put $C = V$, and V will be a function of x and y which additionally contains constants of the differential equation. Then differentiate this equation $C = V$, and so $0 = dV$ will result. And now it is necessary that dV has the propounded form itself as a divisor. Therefore, let it be

$$dV = L(Mdx + Ndy),$$

and L will be the appropriate multiplicator rendering the propounded differential equation integrable. Further, since, while Z denotes any function of V, the formula

$$ZdV = LZ(Mdx + Ndy)$$

is also integrable, the expression LZ will include all multiplicators, by which the propounded differential equation $Mdx + Ndy = 0$ becomes integrable.

Corollary 1

§49 Therefore, if the complete integral of the differential equation $Mdx + Ndy = 0$ can be assigned, not only one but completely all multiplicators rendering the equation integrable can be defined.

Corollary 2

§50 Therefore, because the complete integrals of many differential equations were found by other methods, hence the method treated up to now, which was still applied only to two classes of equations, can be amplified significantly.

SCHOLIUM

§51 Nevertheless, if we do not want to descend to very special cases, the differential equations, whose complete integrals can be assigned, are reduced to a tiny number. And first the differential equations of first degree contained in this form

$$dx(\alpha+\beta x+\gamma y)+dy(\delta+\varepsilon x+\zeta y)=0,$$

since which are easily reduced to homogeneous ones, can also be treated by this method of multiplicators. Furthermore, this form is remarkable

$$dy+Pydx+Qyydx=Rdx;$$

if one single satisfying value of it is known, from it the complete integral can be found, whence in these cases it will be possible to assign appropriate multiplicators. Thirdly, also the cases of this equation, in which it admits the separation of variables, deserve it to be considered

$$dy+yydx=ax^{m}dx;$$

this equation is called the Riccati equation after its discoverer. Finally, there are the cases of this equation

$$ydy+Pydx=Qdx,$$

since which are integrable, they are accommodated to the investigation of multiplicators. Hence this will open a new way to find the equation from a given form of the multiplicators, which equations by means of them become integrable, whence it might be possible to make progress in Analysis not to be hoped for.

PROBLEM 8

§52 Having propounded the differential equation of first order:

$$(\alpha+\beta x+\gamma y)dx+(\delta+\varepsilon x+\zeta y)dy=0,$$

to find the multiplicators rendering it integrable.

Solution

Make this equation homogeneous by putting:

$$x = t + f \quad \text{and} \quad y = u + g,$$

that this expression results

$$(\alpha + \beta f + \gamma g + \beta\gamma u)dt + (\delta + \varepsilon f + \zeta g + \varepsilon t + \zeta u)du = 0$$

which having put

$$\alpha + \beta f + \gamma g = 0 \quad \text{and} \quad \delta + \varepsilon f + \zeta g = 0,$$

whence the quantities f and g are determined, becomes homogeneous, of course,

$$(\beta t + \gamma u)dt + (\varepsilon t + \zeta u)du = 0;$$

and hence it is rendered by means of the multiplicator

$$\frac{1}{\beta tt + (\gamma + \varepsilon)tu + \zeta uu}$$

integrable. Hence having found the letters f and g the propounded equation will become integrable, if it is divided by

$$\beta(x - f)^2 + (\gamma + \varepsilon)(x - f)(y - g) + \zeta(y - g)^2,$$

or by

$$\beta xx + (\gamma + \varepsilon)xy + \zeta yy - (2\beta f + \gamma g + \varepsilon g)x - (2\zeta g + \gamma f + \varepsilon f)y$$
$$+ \beta ff + (\gamma + \varepsilon)fg + \zeta gg.$$

But because it is

$$f = \frac{\alpha\zeta - \gamma\delta}{\gamma\varepsilon - \beta\zeta} \quad \text{and} \quad g = \frac{\beta\delta - \alpha\varepsilon}{\gamma\varepsilon - \beta\zeta},$$

the divisor in question will result as:

$$\beta xx + (\gamma + \varepsilon)xy + \zeta yy + \frac{\alpha\gamma\delta - \alpha\alpha\zeta + \alpha + \delta\varepsilon - \beta\delta\delta}{\gamma\varepsilon - \beta\zeta}$$

$$+\frac{-2\alpha\beta\zeta+\beta\gamma\delta-\beta\delta\varepsilon+\alpha\gamma\varepsilon+\alpha\varepsilon\varepsilon}{\gamma\varepsilon-\beta\zeta}x+\frac{-2\beta\delta\zeta+\alpha\varepsilon\zeta-\alpha\gamma\zeta+\gamma\delta\varepsilon+\gamma\gamma\delta}{\gamma\varepsilon-\beta\zeta}y.$$

But having found one divisor or multiplicator, from it one will easily find all possible ones.

COROLLARY 1

§53 Therefore, the form of the divisor rendering the differential equation

$$(\alpha+\beta x+\gamma y)dx+(\delta+\varepsilon x+\zeta y)dy=0$$

integrable, is

$$\beta xx+(\gamma+\varepsilon)yx+\zeta yy+Ax+By+C,$$

where the constants A, B, C were defined above.

COROLLARY 2

§54 Since the found divisor also satisfies, if it is multiplied by $\gamma\varepsilon-\beta\zeta$, it is plain that in the case, in which it is $\beta\zeta=\gamma\varepsilon$, the divisor will be

$$(\alpha\varepsilon\varepsilon-\beta\delta\varepsilon+\beta\gamma\delta-\alpha\beta\zeta)x+(\gamma\gamma\delta-\alpha\gamma\delta-\alpha\gamma\zeta+\alpha\varepsilon\zeta-\beta\delta\zeta)y+\alpha\gamma\delta-\alpha\alpha\zeta+\alpha\delta\varepsilon-\beta\delta\delta$$

which having put

$$\beta=mf,\quad \gamma=nf,\quad \varepsilon=mg,\quad \zeta=ng,$$

goes over into

$$m(\alpha g-\delta f)(mg-nf)x+n(\alpha g-\delta f)(mg-nf)y+(\alpha g-\delta f)(\delta m-\alpha n).$$

COROLLARY 3

§55 Hence, if the propounded equation was of this kind:

$$[\alpha+f(mx+ny)]dx+[\delta+g(mx+ny)]dy=0,$$

it will be rendered integrable, if it is divided by

$$(mg - nf)(mx + ny) + \delta m - \alpha n$$

or by

$$mx + ny + \frac{\delta m - \alpha n}{mg - nf}.$$

But if it was $mg - nf = 0$, the propounded equation itself is already integrable.

PROBLEM 9

§56 Having propounded this differential equation:

$$dy + Pydx + Qyydx + Rdx = 0,$$

where P, Q and R are functions of x only, if it is known that the equation $y = v$ is satisfied, where v is a function of x, to find multiplicators rendering this equation integrable.

SOLUTION

Since the value $y = v$ satisfies the equation, it will be

$$dy + Pvdx + Qvvdx + Rdx = 0;$$

therefore, if one puts $y = v + \frac{1}{z}$, one will have

$$-\frac{dz}{zz} + \frac{Pdx}{z} + \frac{2Qvdx}{z} + \frac{Qdx}{zz} = 0$$

or

$$dz - (P - 2Qv)zdx - Qdx = 0,$$

which is rendered integrable by means of the multiplicator

$$e^{-\int (P+2Qv)dx}.$$

Therefore, this multiplicator multiplied by zz will satisfy the propounded equation. Therefore, because $z = \frac{1}{y-v}$ is a multiplicator rendering the propounded equation integrable, it will be:

$$\frac{1}{(y-v)^2} e^{-\int (P+2Qv)dx}.$$

For the sake of brevity let it be

$$e^{-\int (P+2Qv)dx} = S.$$

Since the integral of the equation

$$dz - (P+2Qv)zdx - Qdx = 0$$

is

$$Sz - \int QSdx = \text{Const.},$$

all multiplicators in question will be contained in this form:

$$\frac{S}{(y-v)^2} \text{funct.} \left(\frac{S}{y-v} - \int QSdx \right),$$

where by assumption v is a known function of x, and hence also $S = e^{-\int (P+2Qv)dx}$.

COROLLARY 1

§57 Therefore, the multiplicator, which reveals itself at first, is

$$\frac{S}{(y-v)^2},$$

then also

$$\frac{S}{S(y-v) - (y-v)^2 \int QSdx}$$

will be a multiplicator; and even if this multiplicator contains the integral formula $\int QSdx$, it can often become simpler than that one.

COROLLARY 2

§58 For, if S is an exponential quantity, it can happen that $\int QSdx$ takes a form of this kind ST, where T is an algebraic function, in which case the multiplicator will be

$$\frac{1}{y-v-(y-v)^2T}=\frac{1}{(y-v)(1-Ty+Tv)}$$

and hence algebraic, which cannot happen in the first form.

COROLLARY 3

§59 Because in these two cases the multiplicator is a fraction and the variable y enters only its denominator, and there does not ascend higher than a square, innumerable other multiplicators of this kind can be exhibited: For, let $\int QSdx=V$, and it will be possible to multiply the denominator of the fraction $\frac{S}{(y-v)^2}$ by

$$A+B\left(\frac{S}{y-v}-V\right)+C\left(\frac{S}{y-v}\right)^2,$$

and so a more general form of the multiplicator will be:

$$\frac{S}{A(y-v)^2+BS(y-v)-BV(y-v)^2+CSS-2CSV(y-v)+CVV(y-v)^2}$$

or:

$$\frac{S}{(A-BV+CVV)y^2-(2Av-BS-2BVv+2CSV+2CVVv)y+Avv-BSv-BSv-BVvv+CSS+2CSVv+CV^2v^2}.$$

COROLLARY 4

§60 Therefore, if this formula

$$\frac{dy+Pydx+Qyydx+Rdx}{Lyy+My+N}$$

was integrable, the denominator must be of such a nature that it is

$$SL = A - BV + CVV, \quad SM = S(B - 2CV) - 2v(A - BV + CVV)$$

and

$$SN = CSS - Sv(B - 2CV) + vv(A - BV + CVV)$$

where

$$dv + Pvdx + Qvvdx + Rdx = 0, \quad S = e^{-\int (P+2Qv)dx}$$

and $V = \int QSdx$.

Problem 10

§61 Having propounded the preceding differential equation:

$$dy + Pydx + Qyydx + Rdx = 0$$

to find functions L, M and N of x that it divided by the formula

$$Lyy + My + N$$

becomes integrable.

Solution

Therefore, because this formula must be integrable:

$$\frac{dy + dx(Py + Qyy + R)}{Lyy + My + N},$$

it is necessary by means of the general property, after we multiplied by

$$(Lyy + My + N)^2,$$

that it is:

$$-\frac{yydL}{dx} - \frac{ydM}{dx} - \frac{dN}{dx} = \begin{matrix} + QMyy - 2RLy + NP \\ - PLyy + 2QNy - RM \end{matrix}$$

Hence for the determination of the functions L, M and N we obtain these equations:

$$\begin{aligned}
\text{I.} \quad & dL = PLdx - QMdx \\
\text{II.} \quad & dM = 2RLdx - 2QNdx \\
\text{III.} \quad & dN = RMdx - PNdx;
\end{aligned}$$

from the first of them we deduce:

$$M = \frac{PL}{Q} - \frac{dL}{Qdx}$$

and from the second:

$$N = \frac{RL}{Q} - \frac{dM}{2Qdx},$$

which values substituted for M and N in the third give:

$$dN = \frac{PdM}{2Q} - \frac{RdL}{Q}.$$

But because, having assumed the differential dx to be constant, it is

$$dM = \frac{PdL + LdP}{Q} - \frac{PLdQ}{QQ} - \frac{ddL}{Qdx} + \frac{dQdL}{QQdx},$$

it will be

$$N = \frac{RL}{Q} - \frac{PdL}{2QQdx} - \frac{LdP}{2QQdx} + \frac{PLdQ}{2Q^3dx} + \frac{ddL}{2QQdx^2} - \frac{dQdL}{2Q^3dx^2}$$

and

$$dN = \frac{PPdL}{2QQ} + \frac{PLdP}{2QQ} - \frac{PPLdQ}{2Q^3} - \frac{PddL}{2QQdx} + \frac{PdQdL}{2Q^3dx} - \frac{RdL}{Q},$$

which therefore must be equal to the differential of the latter, whence it is

$$\begin{aligned}
0 = {} & QQd^3L - 3QdQddL - PPQQdLdx^2 - 2QQdPdLdx \\
& + 3dQ^2dL + 2PQdQdLdx - QdLddQ + 4Q^3RdLdx^2 \\
& - PQQLdPdx^2 + PPQLdQdx^2 - QQLdxddP + PQLdxddQ \\
& + 3QLdPdQdx - 3PLdQ^2dx + 2Q^3LdRdx^2 - 2Q^2RLdQdx^2.
\end{aligned}$$

But if this equation is multiplied by $\frac{L}{Q^4}$, it can be integrated, and its integral will be

$$\text{Const.} = \frac{LddL}{QQ} - \frac{LdLdQ}{Q^3} - \frac{dL^2}{2QQ} - \frac{PPLLdx^2}{2QQ} - \frac{LLdPdx}{QQ} + \frac{PLLdQdx}{Q^3} + \frac{2RLLdx^2}{Q},$$

which goes over into this form:

$$2EQ^3dx^2 = 2QLddL - 2LLdQ - QdL^2 - PPQLLdx^2 - 2QLLdPdx$$
$$+2PLLdQdx + 4QQRLLdx^2.$$

If one puts $L = zz$, the equation will obtain this form:

$$\frac{2EQ^3dx^2}{z^3} = 4Qddz - 4dQdz - z(PPQdx^2 + 2QdPdx - 2PdQdx - 4QQRdx^2).$$

Corollary 1

§62 Therefore, if by means of the preceding problem the value of L can be assigned, the differential equation of third order found here and the one of second order, to which I reduced the latter, can be solved in general: This resolution, because it would be most difficult otherwise, is to be noted.

Corollary 2

§63 If v was a function of x of such a kind which, put instead of y, satisfies the equation

$$dy + Pydx + Qyydx + Rdx = 0,$$

take

$$S = e^{-\int (P+2Qv)dx}$$

and set $V = \int QSdx$; having done this for our differential equation of third order it will be

$$L = \frac{A - BV + CVV}{S};$$

since this value contains three arbitrary constants, it will therefore be the complete integral of that equation.

Corollary 3

§63 If it is $P = 0$, $Q = 1$ and R any arbitrary function of x, the differential equation of third order will obtain the form:

$$0 = d^3L + 4RdLdx^2 + 2LdRdx^2;$$

in order to find its complete integral at first find a function of x which we want to put $= v$ and to satisfy this equation

$$dy + vvdx + Rdx = 0;$$

then put

$$V = \int e^{-2\int vdx}dx,$$

and it will be

$$L = (A - BV + CVV)e^{+2\int vdx}.$$

Corollary 4

§64 Therefore, the same integral will satisfy this differential equation of second order:

$$2Edx^2 = 2LddL - dL^2 + 4RLLdx^2$$

and, having put $L = zz$, also this one:

$$\frac{Edx^2}{2z^3} = ddz + Rzdx^2,$$

for which it therefore is

$$z = e^{+\int vdx}\sqrt{A - BV + CVV}.$$

Scholium

§65 Therefore, this integration, which can hardly be achieved from other principles, completely deserves some consideration. Hence we obtain the complete integration of the following far-extending differential equation second order:

$$ddz + Sdxdz + Tzdx^2 = \frac{Edx^2}{z^3}e^{-2\int Sdx}.$$

For, first find the value of v from this differential equation of first degree;

$$dv + vvdx + Svdx + Tdx = 0,$$

having found which therefore for the sake of brevity put

$$V = \int e^{-2\int vdx - \int Sdx}dx$$

and it will be

$$z = e^{\int vdx}\sqrt{A + BV + CVV},$$

if only the arbitrary constants A, B, C are taken in such a way that it is

$$AC - \frac{1}{4}BB = E,$$

and so still two constants are arbitrary, as the nature of a complete integration requires it.

EXAMPLE 1

§66 *Let this differential equation be propounded*

$$dy + ydx + yydx - \frac{dx}{x} = 0,$$

whose multiplicators rendering it integrable, are to be investigated.

Therefore, by applying the results of Problem 9 to this, it will be

$$P = 1, \quad Q = 1, \quad \text{and} \quad R = -\frac{1}{x},$$

and since the value $y = \frac{1}{x}$ satisfies this equation, it will be $v = \frac{1}{x}$. Hence it will be

$$S = e^{-\int\left(1+\frac{2}{x}\right)dx} = \frac{1}{xx}e^{-x}$$

and one will have the multiplicator, which reveals itself at first,

$$= e^{-x} \frac{1}{(xy-1)^2}.$$

But furthermore, it is possible to multiply this by any arbitrary function of this form

$$e^{-x} \frac{1}{x(xy-1)} - \int e^{-x} \frac{dx}{xx};$$

but because this formula cannot be integrated, no other appropriate multiplicators can be assigned. Therefore, because of the first this formula is integrable:

$$e^{-x} \frac{1}{(xy-1)^2} \left(dy + ydx + yydx - \frac{dx}{x} \right),$$

whose integral, if x is assumed to be constant, is

$$\frac{-e^x}{x(xy-1)} + X,$$

which differentiated, having put y to be constant, yields

$$\frac{e^{-x} dx (xxy + 2xy - x - 1)}{xx(xy-1)^2} + dX,$$

which must become equal to the other term

$$\frac{e^{-x}}{(xy-1)^2} \left(ydx + yydx - \frac{dx}{x} \right),$$

whence it is

$$dX = \frac{e^{-x} dx}{xx(xy-1)^2} (xxyy - 2xy + 1) = e^{-x} \frac{dx}{xx};$$

and so the complete integral of our equation is

$$\frac{-e^{-x}}{x(xy-1)} + \int e^{-x} \frac{dx}{xx} = \text{Const.}$$

EXAMPLE 2

§67 *To find suitable multiplicators rendering this equation integrable:*

$$dy + yydx - \frac{adx}{(\alpha + \beta x + \gamma xx)^2} = 0.$$

A singular case satisfying this equation is

$$y = \frac{k + \gamma x}{\alpha + \beta x + \gamma xx} = v$$

while

$$k = \frac{1}{2}\beta \pm \sqrt{\frac{1}{4}\beta\beta - \alpha\gamma + a}.$$

Because now it is $P = 0$ and $Q = 1$, it will be

$$S = e^{-\int \frac{2kdx + 2\gamma xdx}{\alpha + \beta x + \gamma xx}}$$

or having put for the sake of brevity

$$\pm\sqrt{\frac{1}{4}\beta\beta - \alpha\gamma + a} = \frac{1}{2}n$$

it will be

$$S = \frac{1}{\alpha + \beta x + \gamma xx} e^{-\int \frac{ndx}{\alpha + \beta x + \gamma xx}}$$

and

$$\int Sdx = -\frac{1}{n} e^{-\int \frac{ndx}{\alpha + \beta x + \gamma xx}}.$$

Therefore, the multiplicator found first is

$$e^{-\int \frac{ndx}{\alpha + \beta x + \gamma xx}} \cdot \frac{\alpha + \beta x + \gamma xx}{((\alpha + \beta x + \gamma xx)y - k - \gamma x)^2},$$

which can further be multiplied by any arbitrary function of this kind

$$e^{-\int \frac{ndx}{\alpha + \beta x + \gamma xx}} \left(\frac{1}{(\alpha + \beta x + \gamma xx)y - k - \gamma x} + \frac{1}{n} \right).$$

Therefore, multiply it by

$$e^{\int \frac{ndx}{\alpha+\beta x+\gamma xx}} \cdot \frac{(\alpha+\beta x+\gamma xx)y-k-\gamma x}{(\alpha+\beta x+\gamma xx)y+n-k-\gamma x}$$

and this algebraic multiplicator will result:

$$\frac{\alpha+\beta x+\gamma xx}{((\alpha+\beta x+\gamma xx)y-k-\gamma x)((\alpha+\beta x+\gamma xx)y+n-k-\gamma x)},$$

which is reduced to this form:

$$\frac{1}{(\alpha+\beta x+\gamma xx)\left(y-\frac{2\gamma x+\beta+\sqrt{\beta\beta-4\alpha\gamma+4\alpha}}{2(\alpha+\beta x+\gamma xx)}\right)\left(y+\frac{2\gamma x+\beta+\sqrt{\beta\beta-4\alpha\gamma+4\alpha}}{2(\alpha+\beta x+\gamma xx)}\right)}.$$

But the complete integral of the equation is

$$e^{-\int \frac{ndx}{\alpha+\beta x+\gamma xx}} \frac{(\alpha+\beta x+\gamma xx)y+n-k-\gamma x}{(\alpha+\beta x+\gamma xx)y-k-\gamma x} = \text{Const.},$$

while $n=\sqrt{\beta\beta-4\alpha\gamma+4a}$ and $k=\frac{\beta+n}{2}$.

Hence the complete integral equation will be

$$e^{-\int \frac{ndx}{\alpha+\beta x+\gamma xx}} \cdot \frac{2(\alpha+\beta x+\gamma xx)y+n-\beta-2\gamma x}{2(\alpha+\beta x+\gamma xx)y-n-\beta-2\gamma x} = \text{Const.},$$

whose nature is obvious, as long as

$$n=\sqrt{\beta\beta-4\alpha\gamma+4a}$$

is a real number.

But if the value of n is imaginary, say $n=m\sqrt{-1}$, because of

$$e^{p\sqrt{-1}} = \cos p + \sqrt{-1}\sin p,$$

the integral can be expressed without imaginary quantities this way. Let

$$-\int \frac{dx}{\alpha+\beta x+\gamma xx} = p \quad \text{and} \quad 2(\alpha+\beta x+\gamma xx)y-\beta-2\gamma x = q,$$

and it will be:

$$(\cos p + \sqrt{-1}\sin p) \cdot \frac{q + m\sqrt{-1}}{q - m\sqrt{-1}} = \text{Const.} = A + B\sqrt{-1},$$

hence it is:

$$q\cos p - m\sin p + (m\cos p + q\sin p)\sqrt{-1} = AQ + Bm + (Bq - Am)\sqrt{-1},$$

equate the real and imaginary terms separately:

$$q\cos p -, \sin p = Aq + Bm, \quad m\cos p + q\sin p = Bq - Am,$$

which two equations agree, if one takes $AA + BB = 1$. Therefore, let the arbitrary constant be $A = \cos\theta$ that it is $B = \sin\theta$ and in the case in which it is $\sqrt{\beta\beta - 4\alpha\gamma + 4a} = m\sqrt{-1}$, the real equation will be

$$q\cos p - m\sin p = q\cos\theta + m\sin\theta \quad \text{or} \quad q = \frac{m(\sin p + \sin\theta)}{\cos p - \cos\theta} = m\cot\frac{\theta - p}{2}.$$

Hence the complete integral equation of the differential equation

$$dy + yydx + \frac{(mm + \beta\beta - 4\alpha\gamma)dx}{4(\alpha + \beta x + \gamma xx)^2} = 0,$$

having put

$$p = \int \frac{-mdx}{\alpha + \beta x + \gamma xx},$$

is

$$2(\alpha + \beta x + \gamma xx)y = \beta + 2\gamma x + m\cot\frac{\theta - p}{2}$$

or

$$y = \frac{\frac{1}{2}\beta + \gamma x + \frac{1}{2}m\cot\frac{\theta - p}{2}}{\alpha + \beta x + \gamma xx},$$

or let $\theta = 180^\circ - \zeta$, and one will have

$$y = \frac{\frac{1}{2}\beta + \gamma x + \frac{1}{2}m\tan\frac{\zeta + p}{2}}{\alpha + \beta x + \gamma xx}.$$

But in this case it is to be noted that the special integral from which we deduced all this becomes imaginary, what is nevertheless no obstruction that hence the complete integral can be exhibited in a real form.

Example 3

§68 *Having propounded the Riccati equation*

$$dy + yydx - ax^m dx = 0,$$

to find suitable multiplicators for the cases of the exponent m, in which it can be separated.

Let $y = v$ be the satisfying value, and because it is

$$P = 0, \quad Q = 1 \quad \text{and} \quad R = -ax^m,$$

the first multiplicator rendering the equation integrable will be

$$e^{-2\int vdx} \frac{1}{(y-v)^2};$$

if it is multiplied by this, the complete integral becomes

$$e^{-2\int vdx} \frac{1}{y-v} - \int e^{-2\int vdx} dx = \text{Const.}$$

Hence, if Z denotes any arbitrary function of this quantity, all multiplicators will be contained in this form:

$$e^{-2\int vdx} \frac{Z}{(y-v)^2}.$$

Hence, if one puts

$$\int e^{-2\int vdx} dx = V,$$

all multiplicators contained in this form

$$\frac{1}{Lyy + My + N}$$

will be obtained [§ 60], if one takes:

$$
\begin{aligned}
L &= e^{2\int v dx}(A - BV + CVV)\\
M &= B - 2CV - 2ve^{2\int v dx}(A - BV + CVV)\\
N &= Ce^{-2\int v dx} - v(B - 2CV) + vve^{2\int v dx}(A - BV + CVV).
\end{aligned}
$$

But this value of L at the same time is the complete integral of this differential equation of third order:

$$0 = d^3L - 4ax^m dLdx^2 - 2maLx^{m-1}dx^3$$

and hence also of this one of second order:

$$Edx^2 = 2LddL - dL^2 - 4aLLx^m dx^2$$

while

$$E = 4AC - BB.$$

SCHOLIUM

§69 Having studied the subject with more attention I even resolved the differential equation of third order by a direct method and detected that the same complete integral of it, which was assigned here, can be found. For, let this equation be propounded:

$$d^3L + 4RdLdx^2 + 2LdRdx^2 = 0,$$

where R is an arbitrary function of x, haven taken the differential dx to be constant. Now, I ask for a function of x, multiplied by which this differential equation becomes integrable. Let S be this function, and the integral of the equation

$$Sd^3L + 4SRdLdx^2 + 2SLdRdx^2 = 0$$

will be

$$SddL - dSdL + L(ddS + 4SRdx^2) = 2Cdx^2,$$

if it is

$$d^3S + 2SdRdx^2 + 4RdSdx^2 = 0.$$

It suffices, of course, to have taken a certain particularly satisfying value. But this equation multiplied by S having neglected the constant gives the integral:

$$SddS - \frac{1}{2}dS^2 + 2SSRdx^2 = 0.$$

Put $S = e^{2\int vdx}$, and it will be

$$2dv + 2vvdx + 2Rdx = 0,$$

whence the task reduces to this that for v at least a particular value is investigated, which satisfies this differential equation of first order:

$$dv + vvdx + Rdx = 0,$$

which I therefore assume as possible. Hence our equation integrated one time, because of $S = e^{2\int vdx}$, will be

$$ddL - 2vdxdL + L(2dvdx + 4vvdx^2 + 4Rdx^2) = 2Ce^{-2\int vdx}dx^2.$$

Therefore, since because of

$$Rdx = -dv - vvdx$$

we have

$$ddL - 2vdxdL - 2Ldxdv = 2Ce^{-2\int vdx}dx^2,$$

its integral obviously is:

$$dL - 2Lvdx = Bdx + 2Cdx\int e^{-2\int vdx}dx$$

and by multiplying the integral again by $e^{-2\int vdx}$ it will result

$$e^{-2\int vdx}L = A + B\int e^{-2\int vdx}dx + 2C\int e^{-2\int vdx}dx\int e^{-2\int vdx}dx.$$

Hence, if for the sake of brevity one puts $\int e^{-2\int vdx}dx = V$, we will have

$$L = e^{2\int vdx}(A + BV + 2CVV)$$

completely as we found before.

PROBLEM 11

§70 Having propounded the Riccati equation

$$dy + yydx = ax^m dx$$

to find its particular integrals in the cases in which it is separable.

SOLUTION

By putting $a = cc$ and $m = -4n$, attribute this form to the equation:

$$dy + yydx - ccx^{-4n}dx = 0.$$

For, since the question is about particular integrals, it is not important, whether they are real or not. But to find these cases, in which y can be expressed by means of a function of x, in an easier way and in one operation let us set

$$y = cx^{-2n} + \frac{dz}{zdx}$$

and having assumed dx to be constant, we will obtain this differential equation of second order:

$$-2ncx^{-2n-1}dx + \frac{ddz}{zdx} + \frac{2cx^{-2n}dz}{z} = 0,$$

or

$$\frac{ddz}{dx^2} + \frac{2cdz}{x^{2n}dx} - \frac{2ncz}{x^{2n+1}} = 0,$$

whose value we want to assume to be:

$$z = Ax^n + Bx^{3n-1} + Cx^{5n-2} + Dx^{7n-3} + Ex^{9n-4} + \text{etc.};$$

after having substituted this value in the correct way we will obtain:

$$\begin{array}{llllll}
 & 0 = \; n(n-1)Ax^{n-2} & + \; (3n-1)(3n-2)Bx^{3n-3} & + \; (5n-2)(5n-3)Cx^{5n-4} & + \; \text{etc.} & \\
+ \; 2ncAx^{-n-1} & + \; 2(3n-1)cB & + \; 2(5n-2)cC & + \; 2(7n-3)cD & & \\
- \; 2ncA & - \; 2ncB & \quad\;\; 2ncC & - \; 2ncD & &
\end{array}$$

whence the assumed coefficients are determined this way:

$$\begin{aligned}
&2(2n-1)cB + n(n-1)A = 0, \qquad && B = \frac{-n(n-1)A}{2(2n-1)c} \\
&2(4n-1)cC + (3n-1)(3n-2)B = 0, && C = \frac{-(3n-1)(3n-2)B}{4(2n-1)c} \\
&2(6n-3)cD + (5n-2)(5n-3)C = 0, && D = \frac{-(5n-2)(5n-3)C}{6(2n-1)} \\
& && \text{etc.}
\end{aligned}$$

Therefore, if one coefficient vanishes, all following ones also vanish, what happens in these cases:

$$\begin{aligned}
&n = 0, \quad n = \frac{1}{3}, \quad n = \frac{2}{5}, \quad n = \frac{3}{7}, \quad \text{etc.} \\
&n = 1, \quad n = \frac{2}{3}, \quad n = \frac{3}{5}, \quad n = \frac{4}{7}, \quad \text{etc.}
\end{aligned}$$

Therefore, if, while i denotes any integer, it was

$$n = \frac{i}{2i \pm i},$$

the resolution of the equation can be exhibited. For, it will be

$$y = c^{-2n} + \frac{dz}{zdx},$$

where

$$z = Ax^n + Bx^{2n-1} + Cx^{5n-2} + Dx^{7n-3} + Ex^{9n-4} + \text{etc.}$$

Therefore, this particular value of y will result:

$$y = cx^{-2n} + \frac{nAx^{n-1} + (3n-1)Bx^{3n-1} + (5n-2)Cx^{5n-3} + \text{etc.}}{Ax^n + Bx^{3n-1} + Cx^{5n-2} + \text{etc.}}$$

COROLLARY 1

§71 Therefore, if this particular value of y is called $= v$, a suitable multiplicator of the propounded equation will be

$$= e^{-2\int vdx} \cdot \frac{1}{(y-v)^2}.$$

And if one puts

$$\int e^{-2\int vdx} dx = V,$$

having taken $A = 0$ and $C = 0$, another simpler factor will be [§ 68]

$$\frac{1}{e^{2\int vdx}Vyy - (1 + 2ce^{2\int vdx}V)y + v + vve^{2\int vdx}V}.$$

COROLLARY 2

§72 But it is

$$\int vdx = \frac{-c}{(2n-1)x^{2n-1}} + \log(Ax^n + Bx^{2n-1} + Cx^{5n-1} + \text{etc.}),$$

whence it is

$$e^{-2\int vdx} = e^{\frac{2n}{(2n-1)}x^{2n-1}} \frac{1}{(Ax^n + Bx^{3n-1} + Cx^{5n-2} + \text{etc.})^2};$$

Hence one will further find the value of

$$V = \int e^{-2\int vdx} dx.$$

If it was of this kind

$$e^{-2\int vdx} T,$$

while T is an algebraic function, the above multiplicator will algebraic.

COROLLARY 3

§73 Having found the value v, or a particular integral of the propounded equation, hence one will immediately have the complete integral of the same, which will be:

$$\frac{e^{-2\int vdx}}{y-v} - \int e^{-2\int vdx}dx = \text{Const.}$$

CASE 1 IN WHICH IT IS $n = 0$

§74 Therefore, for this equation

$$dy + yydx = ccdx,$$

because of $B = 0$, $C = 0$ etc. a particular value will be $y = c$. Hence having put $v = c$, it will be

$$e^{-2\int vdx} = e^{-2cx} \quad \text{and} \quad V = \int e^{-2\int vdx}dx = -\frac{1}{2c}e^{-2cx};$$

hence the complete integral is

$$\frac{e^{-2cx}}{y-c} + \frac{y}{2c}e^{-2cx} = \text{Const.}.$$

or

$$\frac{e^{-2cx(y+c)}}{y-c} = \text{Const.}$$

Further, because of

$$e^{2\int vdx}V = -\frac{1}{2c} \quad \text{and} \quad v = c,$$

an algebraic multiplicator will be:

$$\frac{1}{-\frac{1}{2c}yy + \frac{1}{2}c},$$

which is reduced to

$$\frac{1}{yy - cc}$$

as it is perspicuous per se.

§75 Therefore, for this equation

$$dy + yydx = \frac{ccdx}{x^4}$$

because of $B = 0$, $C = 0$ a particular value will be

$$y = \frac{c}{xx} + \frac{1}{x}.$$

Hence having put

$$v = \frac{c}{xx} + \frac{1}{x},$$

it will be

$$e^{-2\int vdx} = \frac{e^{\frac{2c}{x}}}{xx} \quad \text{and} \quad V = -\frac{1}{2c}e^{\frac{2c}{x}}.$$

Hence the complete integral is

$$\frac{e^{\frac{2c}{x}}}{xxy - x - c} + \frac{e^{\frac{2c}{x}}}{2c} = \text{Const.}$$

or

$$e^{\frac{2c}{x}} \cdot \frac{xxy - x + c}{xxy - x - c} = \text{Const.}$$

Further, because of

$$e^{2\int vdx}V = -\frac{xx}{2c} \quad \text{and} \quad v = \frac{x + c}{xx},$$

one will have the algebraic multiplicator:

$$\frac{1}{xxyy - 2xy + 1 - \frac{cc}{xx}} = \frac{1}{(xy-1)^2 - \frac{cc}{xx}}$$

or the propounded equation

$$dy + yydx - \frac{ccdx}{x^4} = 0$$

becomes integrable, if it is divided by

$$(xy-1)^2 - \frac{cc}{xx}.$$

CASE 3 IN WHICH IT IS $n = \frac{1}{3}$

§76 Therefore, for this equation

$$dy + yydx - ccx^{-\frac{4}{3}}dx = 0$$

it is $B = -\frac{A}{3c}$, $C = 0$ etc. whence a particular integral is

$$y = cx^{-\frac{2}{3}} + \frac{cx^{-\frac{2}{3}}}{3cx^{-\frac{1}{3}} - 1} = \frac{3ccx^{-\frac{1}{3}}}{3cx^{\frac{1}{3}} - 1} = v$$

and

$$e^{-2\int vdx} = e^{-6cx^{\frac{1}{3}}} \frac{\text{Const.}}{\left(x^{\frac{1}{3}-\frac{1}{3c}}\right)^2} = e^{-6cx^{\frac{1}{3}}} \frac{1}{\left(3cx^{\frac{1}{3}} - 1\right)^2}$$

and hence

$$V = \int e^{-6cx^{\frac{1}{3}}} \frac{dx}{\left(3cx^{\frac{1}{3}} - 1\right)^2} = -e^{-6cx^{\frac{1}{3}}} \frac{3cx^{\frac{1}{3}} + 1}{18c^3\left(3cx^{\frac{1}{3}} - 1\right)}.$$

Hence the complete integral is

$$\frac{e^{-6cx^{\frac{1}{3}}}}{\left(3cx^{\frac{1}{3}} - 1\right)^2 y - 3ccx^{-\frac{1}{3}}\left(3cx^{\frac{1}{3}} - 1\right)} + \frac{e^{-6cx^{\frac{1}{3}}}\left(3cx^{\frac{1}{3}} + 1\right)}{18c^4\left(3cx^{\frac{1}{3}} - 1\right)} = \text{Const.}$$

or

$$e^{-6cx^{\frac{1}{3}}} \frac{y\left(1 + 3cx^{\frac{1}{3}}\right) + 3ccx^{-\frac{1}{3}}}{y\left(1 - 3cx^{\frac{1}{3}}\right) - 3ccx^{-\frac{1}{3}}} = \text{Const.}$$

Then, because of

$$e^{2\int vdx}V = \frac{1 - 9ccx^{\frac{2}{3}}}{18c^3},$$

this divisor rendering the equation integrable will be:

$$\left(y + 3ccx^{-\frac{1}{3}}\right)^2 - 9ccx^{\frac{2}{3}}yy.$$

CASE 4 IN WHICH IT IS $n = \frac{2}{3}$

§77 Therefore, for this equation

$$dy + yydx - ccx^{-\frac{8}{3}}dx = 0$$

it is $B = +\frac{A}{3c}$, $C = 0$ etc., whence the particular integral is:

$$y = cx^{-\frac{4}{3}}\frac{2cx^{-\frac{1}{3}}+1}{3cx^{\frac{2}{3}}+x} = \frac{3ccx^{-\frac{2}{3}}+3cx^{-\frac{1}{3}}+1}{3cx^{\frac{2}{3}}+x} = v$$

and

$$e^{-2\int vdx} = e^{6cx^{-\frac{1}{3}}}\cdot\frac{1}{(3cx^{\frac{2}{3}}+x)^2};$$

hence it is further found:

$$V = \int\frac{e^{6cx^{-\frac{1}{3}}}dx}{(3cx^{\frac{2}{3}}+x)^2} = \frac{-e^{6cx^{-\frac{1}{3}}}(3cx^{\frac{2}{3}}-x)}{18c^3(3cx^{\frac{2}{3}}+x)}.$$

Hence the complete integral will be:

$$e^{6cx^{-\frac{1}{3}}}\frac{(x-3cx^{\frac{2}{3}})y-1+3cx^{-\frac{1}{3}}-3ccx^{-\frac{2}{3}}}{(x+3cx^{\frac{2}{3}})y-1-3cx^{-\frac{1}{3}}-3ccx^{-\frac{2}{3}}} = \text{Const.}$$

Then because of

$$e^{2\int vdx}V = \frac{xx-9ccx^{\frac{4}{3}}}{18c^3}$$

an algebraic divisor rendering the propounded equation integrable results as:

$$((x+3cx^{\frac{2}{3}})y-1-3cx^{-\frac{1}{3}}-3ccx^{-\frac{2}{3}})((x+3cx^{\frac{2}{3}})y-1+3cx^{-\frac{1}{3}}-3ccx^{-\frac{2}{3}}).$$

CASE 5 IN WHICH IT IS $n = \frac{2}{5}$

§78 Therefore, for the equation

$$dy + yydx - ccx^{-\frac{8}{5}} = 0$$

it will be

$$B = -\frac{3A}{5c}; \quad C = -\frac{B}{5c} = +\frac{3A}{35cc}; \quad D = 0 \quad \text{etc.}$$

and hence a particular integral:

$$y = cx^{-\frac{4}{5}} + \frac{\frac{2}{5}x^{-\frac{3}{5}} - \frac{1}{5} \cdot \frac{3}{5c}x^{-\frac{4}{5}}}{x^{\frac{2}{5}} - \frac{3}{5c}x^{\frac{1}{5}} + \frac{3}{25cc}} = cx^{-\frac{4}{5}} + \frac{10ccx^{-\frac{3}{4}} - 3cx^{-\frac{4}{5}}}{25ccx^{\frac{2}{5}} - 15cx^{\frac{1}{5}} + 3}$$

or

$$y = \frac{25c^3x^{-\frac{2}{5}} - 5ccx^{-\frac{3}{5}}}{25ccx^{\frac{2}{5}} - 15cx^{\frac{1}{5}} + 3} = v.$$

Hence the complete integral results as:

$$e^{-10cx^{\frac{1}{3}}} \cdot \frac{(3 + 15cx^{\frac{1}{5}} + 25ccx^{\frac{2}{5}})y + 5ccx^{-\frac{3}{5}} + 25c^3x^{-\frac{2}{5}}}{(3 - 15cx^{\frac{1}{5}} + 25ccx^{\frac{2}{5}})y + 5ccx^{-\frac{3}{5}} - 25c^3x^{-\frac{2}{5}}} = \text{Const.}$$

And if in this fraction one puts

the numerator $\quad (3 + 15cx^{\frac{1}{5}} + 25ccx^{\frac{2}{5}})y + 5ccx^{-\frac{3}{5}} + 25c^3x^{-\frac{2}{5}} = P$ and

the denominator $\quad (3 - 15cx^{\frac{1}{5}} + 25ccx^{\frac{2}{5}})y + 5ccx^{-\frac{3}{5}} - 25c^3x^{-\frac{2}{5}} = Q,$

the divisor rendering the propounded equation integrable will be $= PQ$.

CASE 6 IN WHICH IT IS $n = \frac{3}{5}$

§79 Therefore, for this equation

$$dy + yydx - ccx^{-\frac{12}{5}}dx = 0$$

it will be

$$B = \frac{3A}{5c} \quad \text{and} \quad C = \frac{B}{5c} = \frac{3A}{25cc}, \quad D = 0 \quad \text{etc.}$$

and hence the particular integral results to be:

$$y = cx^{-\frac{6}{5}} + \frac{15ccx^{-\frac{2}{5}} + 12cx^{-\frac{1}{5}} + 3}{25ccx^{\frac{3}{5}} + 15cx^{\frac{4}{5}} + 3x}$$

or

$$y=\frac{25c^3x^{-\frac{3}{5}}+30ccx^{-\frac{2}{5}}+15cx^{-\frac{1}{5}}+3}{25ccx^{\frac{3}{5}}+15cx^{\frac{4}{5}}+3x}=v,$$

whence the complete integral is obtained:

$$e^{10cx^{-\frac{1}{5}}}\cdot\frac{(3x-15cx^{\frac{4}{5}}+25ccx^{\frac{3}{5}})y-3+15cx^{-\frac{1}{5}}-30ccx^{-\frac{2}{5}}+25c^3x^{-\frac{3}{5}}}{(3x+15cx^{\frac{4}{5}}+25ccx^{\frac{3}{5}})y-3-15cx^{-\frac{1}{5}}-30ccx^{-\frac{2}{5}}-25c^3x^{-\frac{3}{5}}}=\text{Const.}$$

And having neglected the exponential factor $e^{10cx^{-\frac{1}{3}}}$ the product of the numerator and the denominator will yield the divisor, divided by which the propounded equation becomes integrable.

PROBLEM 12

§80 While i denotes any integer number to exhibit the resolution of this equation:

$$dy+yydx-ccx^{\frac{-4i}{2i+1}}dx=0.$$

SOLUTION

Therefore, because it is $n=\frac{i}{2i+1}$, one will find

$$\begin{aligned}
B &= -\frac{(i+1)i}{2(2i+1)c}A\\
C &= +\frac{(i+2)(i+1)i(i-1)}{2\cdot 4(2i+1)^2c^2}A\\
D &= -\frac{(i+3)(i+2)(i+1)i(i-1)(i-2)}{2\cdot 4\cdot 6(2i+1)^3c^3}A\\
E &= +\frac{(i+4)(i+3)(i+2)(i+1)i(i-1)(i-2)(i-3)}{2\cdot 4\cdot 6\cdot 8(2i+1)^4c^4}A\\
&\text{etc.,}
\end{aligned}$$

on the other hand the particular integral will be:

$$y=cx^{\frac{-2i}{2i+1}}+\frac{\frac{i}{2i+1}Ax^{\frac{-i-1}{2i+1}}+\frac{i-1}{2i+1}Bx^{\frac{-i-2}{2i+1}}+\frac{i-2}{2i+1}Cx^{\frac{-i-3}{2i+1}}+\frac{i-3}{2i+1}Dx^{\frac{-i-4}{2i+1}}+\text{etc.}}{Ax^{\frac{i}{2i+1}}+Bx^{\frac{i-1}{2i+1}}+Cx^{\frac{i-2}{2i+1}}+Dx^{\frac{i-3}{2i+1}}+\text{etc.}}$$

to reduce which to the same denominator, let us set:

$$\begin{aligned}
\mathfrak{A} &= cA \\
\mathfrak{B} &- \frac{i(i-1)}{2(2i+1)}A \\
\mathfrak{C} &+ \frac{(i+1)i(i-1)(i-2)}{2\cdot 4(2i+1)^2c}A \\
\mathfrak{D} &- \frac{(i+2)(i+1)i(i-1)(i-2)(i-3)}{2\cdot 4\cdot 6(2i+1)^3c^2}A \\
&\text{etc.,}
\end{aligned}$$

whence it will be:

$$y = \frac{\mathfrak{A}x^{\frac{-i}{2i+1}} + \mathfrak{B}x^{\frac{-i-1}{2i+1}} + \mathfrak{C}x^{\frac{-i-2}{2i+1}} + \mathfrak{D}x^{\frac{-i-3}{2i+1}}}{Ax^{\frac{i}{2i+1}} + Bx^{\frac{i-1}{2i+1}} + Cx^{\frac{i-2}{2i+1}} + Dx^{\frac{i-3}{2i+1}}}.$$

Further, for the sake of brevity let us put:

$$\begin{aligned}
Ax^{\frac{i}{2i+1}} + Bx^{\frac{i-1}{2i+1}} + Cx^{\frac{i-2}{2i+1}} + Dx^{\frac{i-3}{2i+1}} + \text{etc.} &= P \\
Ax^{\frac{i}{2i+1}} - Bx^{\frac{i-1}{2i+1}} + Cx^{\frac{i-2}{2i+1}} - Dx^{\frac{i-3}{2i+1}} + \text{etc.} &= Q \\
\mathfrak{A}x^{\frac{-i}{2i+1}} + \mathfrak{B}x^{\frac{-i-1}{2i+1}} + \mathfrak{C}x^{\frac{-i-2}{2i+1}} + \mathfrak{D}x^{\frac{-i-3}{2i+1}} + \text{etc.} &= \mathfrak{P} \\
-\mathfrak{A}x^{\frac{-i}{2i+1}} + \mathfrak{B}x^{\frac{-i-1}{2i+1}} - \mathfrak{C}x^{\frac{-i-2}{2i+1}} + \mathfrak{D}x^{\frac{-i-3}{2i+1}} - \text{etc.} &= \mathfrak{Q}
\end{aligned}$$

and the complete integral will be:

$$e^{-2(2i+1)cx^{\frac{+1}{2i+1}}}\frac{Qy-\mathfrak{Q}}{Py-\mathfrak{P}} = \text{Const.}$$

But then the divisor rendering the propounded equation integrable will be $= (Py-\mathfrak{P})(Qy-\mathfrak{Q})$.

Corollary 1

§81 Therefore, if in the equation

$$dy + yydx + \alpha x^{\frac{-4i}{2i+1}}dx = 0$$

the coefficient α was a negative quantity, such that having put $\alpha = -cc$ c is a real quantity, the complete integral found here has a real form, and can be easily exhibited in each case; and likewise one cane exhibit the divisor, which renders the equation integrable.

COROLLARY 2

§82 But if α was a positive quantity, say $\alpha = aa$ that one has this equation:

$$dy + yydx + aax^{\frac{-4i}{2i+1}}dx = 0,$$

it will be $c = a\sqrt{-1}$, and the coefficients B, D, F etc. and $\mathfrak{A}$, $\mathfrak{C}$, $\mathfrak{E}$ will become negative; hence the particular values $y = \frac{\mathfrak{P}}{P}$ and $y = \frac{\mathfrak{Q}}{Q}$ will turn out to be imaginary.

COROLLARY 3

§83 Nevertheless in the case, in which it is $c = a\sqrt{-1}$ and $cc = -aa$, $P + Q$ and $\mathfrak{P} + \mathfrak{Q}$ will become real quantities, but $P - Q$ and $\mathfrak{P} - \mathfrak{Q}$ imaginary ones. Therefore, if one puts

$$P + Q = 2R, \quad P - Q = 2S\sqrt{-1}, \quad \mathfrak{P} + \mathfrak{Q} = 2\mathfrak{R} \quad \text{and} \quad \mathfrak{P} - \mathfrak{Q} = 2\mathfrak{S}\sqrt{-1},$$

R, S, $\mathfrak{R}$ and $\mathfrak{S}$ will be real quantities and because of

$$P = R + S\sqrt{-1}, \quad Q = R - S\sqrt{-1}, \quad \mathfrak{P} = \mathfrak{R} + \mathfrak{S}\sqrt{-1}, \quad \mathfrak{Q} = \mathfrak{R} - \mathfrak{S}\sqrt{-1}$$

the divisor rendering the equation integrable will become

$$(RR + SS)yy - 2(R\mathfrak{R} + S\mathfrak{S})y + \mathfrak{R}\mathfrak{R} + \mathfrak{S}\mathfrak{S}$$

and hence real.

COROLLARY 4

§84 But in the same case $c = a\sqrt{-1}$ because of

$$e^{-p\sqrt{-1}} = \cos p - \sqrt{-1}\sin p,$$

it will be

$$e^{-2(2i+1)\alpha x^{\frac{1}{2i+1}}\sqrt{-1}} = \cos 2(2i+1)\alpha x^{\frac{1}{2i+1}} - \sqrt{-1}\sin 2(2i+1)\alpha x^{\frac{1}{2i+1}}$$

whence having for the sake of brevity put

$$2(2i+1)\alpha x^{\frac{1}{2i+1}} = p,$$

the complete integral will be:

$$(\cos p - \sqrt{-1}\sin p)\cdot\frac{(R+S\sqrt{-1})y - \mathfrak{R} + \mathfrak{S}\sqrt{-1}}{(R-S\sqrt{-1})y - \mathfrak{R} - \mathfrak{S}\sqrt{-1}} = \text{Const.},$$

which form is imaginary.

COROLLARY 5

§85 But attribute such a form to the constant: $\alpha - \beta\sqrt{-1}$, and having expanded the integral equation, it will be:

$$\begin{aligned}(Ry-\mathfrak{R})\cos p - (Ry-\mathfrak{R})\sin p\sqrt{-1} - (Sy-\mathfrak{S})\cos p\sqrt{-1} - (Sy-\mathfrak{S})\sin p\\ = (Ry-\mathfrak{R})\alpha - (Ry-\mathfrak{R})\beta\sqrt{-1} + (Sy-\mathfrak{S})\alpha\sqrt{-1} + (Sy-\mathfrak{S})\beta.\end{aligned}$$

Now equate the real and imaginary parts separately:

$$\begin{aligned}(Ry-\mathfrak{R})\cos p - (Sy-\mathfrak{S})\sin p &= \alpha(Ry-\mathfrak{R}) + \beta(Sy-\mathfrak{S})\\ (Ry-\mathfrak{R})\sin p + (Sy-\mathfrak{S})\cos p &= \beta(Ry-\mathfrak{R}) - \alpha(Sy-\mathfrak{S}),\end{aligned}$$

which two equations agree, if it only is

$$\alpha\alpha + \beta\beta = 1.$$

Therefore, let $\alpha = \cos\zeta$ and $\beta = \sin\zeta$ and from each of them this equation will result

$$\frac{Ry-\mathfrak{R}}{Sy-\mathfrak{S}} = \frac{\sin p + \sin\zeta}{\cos p + \cos\zeta} = \cot\frac{\zeta - p}{2}.$$

COROLLARY 6

§86 Therefore, having taken any angle for ζ, if it is $c = a\sqrt{-1}$, the complete integral of the propounded equation will be

$$\frac{Ry - \mathfrak{R}}{Sy - \mathfrak{S}} = \cot\frac{\zeta - p}{2}$$

or

$$y = \frac{\mathfrak{R}\sin\frac{\zeta-p}{2} - \mathfrak{S}\cos\frac{\zeta-p}{2}}{R\sin\frac{\zeta-p}{2} - S\cos\frac{\zeta-p}{2}}$$

while $p = 2(2i+1)\alpha x^{\frac{1}{2i+1}}$.

PROBLEM 13

§87 While i denotes any arbitrary integer number to exhibit the resolution of this equation:

$$dy + yydx - ccx^{\frac{-4i}{2i-1}}dx = 0.$$

SOLUTION

Since it is $n = \frac{i}{2i-1}$, this resolution can be derived from the solution of the preceding problem by putting $-i$ instead of i. Hence, attribute the following values to the letters B, C, D etc.:

$$\begin{aligned}
B &= +\frac{i(i-1)}{2(2i-1)c}A\\
C &= +\frac{(i+1)i(i-1)(i-2)}{2\cdot 4(2i-1)^2c^2}A\\
D &= +\frac{(i+2)(i+1)i(i-1)(i-2)(i-3)}{2\cdot 4\cdot 6(2i-1)^3c^3}A\\
&\text{etc.}
\end{aligned}$$

But the determination of the other letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ will be as follows:

$$\mathfrak{A} = cA$$
$$\mathfrak{B} = +\frac{(i+1)i}{2(2i-1)}A$$
$$\mathfrak{C} = +\frac{(i+2)(i+1)i(i-1)}{2\cdot 4(2i-1)^2c}A$$
$$\mathfrak{D} = +\frac{(i+3)(i+2)(i+1)i(i-1)(i-2)}{2\cdot 4\cdot 6(2i-1)^3c^2}A$$
etc.

Having constituted these values for the sake of brevity put:

$$Ax^{\frac{+i}{2i-1}} + Bx^{\frac{+i+1}{2i-1}} + Cx^{\frac{+i+2}{2i-1}} + Dx^{\frac{+i+3}{2i-1}} + \text{etc.} = P$$
$$Ax^{\frac{+i}{2i-1}} - Bx^{\frac{+i+1}{2i-1}} + Cx^{\frac{+i+2}{2i-1}} - Dx^{\frac{+i+3}{2i-1}} + \text{etc.} = Q$$
$$\mathfrak{A}x^{\frac{-i}{2i-1}} + \mathfrak{B}x^{\frac{-i+1}{2i-1}} + \mathfrak{C}x^{\frac{-i+2}{2i-1}} + \mathfrak{D}x^{\frac{-i+3}{2i-1}} + \text{etc.} = \mathfrak{P}$$
$$-\mathfrak{A}x^{\frac{-i}{2i-1}} + \mathfrak{B}x^{\frac{-i+1}{2i-1}} - \mathfrak{C}x^{\frac{-i+2}{2i-1}} + \mathfrak{D}x^{\frac{-i+3}{2i-1}} - \text{etc.} = \mathfrak{Q}$$

and hence one immediately has two particular solutions:

$$\text{I.}\quad y = \frac{\mathfrak{P}}{P} \quad \text{and} \quad \text{II.}\quad y = \frac{\mathfrak{Q}}{Q}.$$

But then the complete integral equation will be:

$$e^{2(2i-1)cx^{\frac{-1}{2i-1}}}\frac{Qy-\mathfrak{Q}}{Py-\mathfrak{P}} = \text{Const.}$$

and the divisor rendering the propounded equation integrable will be $= (Py-\mathfrak{P})(Qy-\mathfrak{Q})$.

COROLLARY 1

§88 But if the propounded equation was of this kind:

$$dy + yydx + aax^{\frac{-4i}{2i-1}}dx = 0,$$

that it is $cc = -aa$ and $c = a\sqrt{-1}$, the exhibited particular solution will become imaginary because of the imaginary B, D, F and $\mathfrak{A}$, $\mathfrak{C}$, $\mathfrak{E}$ etc., whereas the values of the remaining letters are real.

COROLLARY 2

§89 But if one puts

$$P+Q=2R,\quad P-Q=2S\sqrt{-1},\quad \mathfrak{P}+\mathfrak{Q}=2\mathfrak{R}\quad\text{and}\quad \mathfrak{P}-\mathfrak{Q}=2\mathfrak{S}\sqrt{-1},$$

the quantities R, S, $\mathfrak{R}$ and $\mathfrak{S}$ will nevertheless, as before, be real, and the divisor rendering the equation integrable will be:

$$(RR+SS)yy-2(R\mathfrak{R}+S\mathfrak{S})y+\mathfrak{R}\mathfrak{R}+\mathfrak{S}\mathfrak{S}.$$

COROLLARY 3

§90 But then, if one for the sake of brevity puts

$$2(2i-1)\alpha x^{\frac{-i}{2i-1}}=p,$$

the complete integral will be:

$$\frac{Ry-\mathfrak{R}}{Sy-\mathfrak{S}}=\cot\frac{\zeta+p}{2},$$

whence one finds:

$$y=\frac{\mathfrak{R}\sin\frac{\zeta+p}{2}-\mathfrak{S}\cos\frac{\zeta+p}{2}}{R\sin\frac{\zeta+p}{2}-S\cos\frac{\zeta+p}{2}}$$

where the angle ζ takes the part of the arbitrary constant.

SCHOLIUM

§91 The solutions of these last two problems were not expanded so by accurate analysis as derived by induction from the particular cases explained above, since the progression from these cases to the following was obvious. But the foundation of these solutions mainly is mainly based on this, that a particular solution, from which all others are deduced is actually a double one, since the quantity c, only whose square occurs in the differential equation, can be taken negatively and positively. But as often as two particular solutions of equations of this kind are known, from them the general solution and hence the multiplicators rendering it integrable can be found a lot easier, which will be worth one's while to have explained it more clearly.

PROBLEM 14

§92 Having found two particular solutions of an equation of this kind:

$$dy + Pydx + Qyydx + Rdx = 0$$

to find its general solution and a multiplicator rendering it integrable.

SOLUTION

Let M and N be functions of x of such a kind, which substituted for y satisfy the propounded equation such that it is:

$$dM + PMdx + QM^2dx + Rdx = 0$$

and

$$dN + PNdx + QN^2dx + Rdx = 0.$$

Put

$$\frac{y-M}{y-N} = z \quad \text{and} \quad y = \frac{M-Nz}{1-z},$$

it will be

$$dy = \frac{dM - zdM + Mdz - Ndz - zdN + zzdN}{(1-z)^2},$$

having substituted these values in the propounded equation and having multiplied the whole equation by $(1-z)^2$ it will result:

$$(1-z)dM - z(1-z)dN + (M-N)dz + P(1-z)Mdx - P(1-z)Nzdx$$

$$+QMMdx - 2QMNzdx + QNNzzdx + R(1-z)^2dx = 0.$$

Now substitute the values to result from the two above differentials for dM and dN:

$$
\begin{aligned}
&-P\ \ (1-z)Mdx - Q(1-z)M^2dx\ -R\ \ (1-z)dx\\
&+Pz(1-z)Ndx\ +Qz(1-z)N^2dx + Rz(1-z)dx + (M-N)dx = 0\\
&+P\ \ (1-z)Mdx + QM^2dx \qquad\quad\ +R\ \ (1-z)^2dx\\
&-Pz(1-z)Ndx\ -2QMNzdx\\
&\qquad\qquad\qquad\quad +QN^2zzdx;
\end{aligned}
$$

having ordered this equation it will result:

$$QzM^2dx + QzN^2dx - 2QMNzdx + (M-N)dz = 0$$

or

$$Q(M-N)dx + \frac{dz}{z} = 0,$$

such that it is:

$$z = Ce^{-\int Q(M-N)dx},$$

whence the general integrated equation will be:

$$e^{\int Q(M-N)dx}\frac{y-M}{y-N} = \text{Const.}$$

But in order to find the multiplicator note that the propounded equation after the substitution was multiplied by $(1-z)^2$ first, but, then having divided it by $z(M-N)$, became integrable. Therefore, having multiplied it by $\frac{(1-z)^2}{(M-N)z}$ it will become integrable immediately: Therefore, the factor will be $\frac{(1-z)^2}{(M-N)z}$, which because of $z = \frac{y-M}{y-N}$ will obtain this form:

$$\frac{M-N}{(y-M)(y-N)}.$$

Problem 15

§93 Having propounded the equation

$$ydy + Pydx + Qdx = 0,$$

to find conditions of the functions P and Q that a multiplicator of this kind $(y+M)^n$ renders it integrable.

Therefore, from the nature of differentials it must be:

$$\frac{1}{dx}d.y(y+M) = \frac{1}{dy}d.(Py+Q)(y+M)^n,$$

because hence M is a function of x only, it will be

$$ny(y+M)^{n-1}\frac{dM}{dx} = P(y+M)^n + n(Py+Q)(y+M)^{n-1},$$

which divided by $(y+M)^{n-1}$ goes over into this one:

$$\frac{nydM}{dx} = (n+1)Py + PM + nQ,$$

whence it is necessary that it is:

$$P = \frac{ndM}{(n+1)dx} \quad \text{and} \quad Q = \frac{-PM}{n} = -\frac{MdM}{(n+1)dx}.$$

Therefore, having substituted these values the equation

$$ydy + \frac{nydM}{n+1} - \frac{MdM}{n+1} = 0$$

becomes integrable, if it is multiplied by $(y+M)^n$.

COROLLARY 1

§94 Since this equation is homogeneous, it is also integrable, if it is divided by

$$(n+1)yy + nyM - MM = (y+M)((n+1)y - M).$$

And therefore hence no new equations which can be treated by this method are obtained.

COROLLARY 2

§95 But since we have these two multiplicators

$$(y+M)^n \quad \text{and} \quad \frac{1}{(y+M)((n+1)y-M)},$$

if the one is divided by the other, the quotient, having equated it to an arbitrary constant, will give the complete integral. Hence the equation

$$ydy + \frac{nydM}{n+1} - \frac{MdM}{n+1} = 0$$

integrated in general yields:

$$(y+M)^{n+1}((n+1)y - M) = \text{Const.}$$

PROBLEM 16

§96 Having propounded the equation

$$ydy + Pydx + Qdx = 0,$$

to find the conditions of the functions P and Q that a multiplicator of this kind

$$(yy + My + N)^n$$

renders it integrable.

SOLUTION

From the nature of differentials it is necessary that it is:

$$\frac{1}{dx}d.y(yy+My+N)^n = \frac{1}{dy}d.(Py+Q)(yy+My+N)^n.$$

Therefore, since M, N, P and Q by assumption are functions of x, having done the expansion it will be:

$$ny(yy+My+N)^{n-1}\left(y + \frac{dM}{dx} + \frac{dN}{dx}\right)$$
$$= P(yy+My+N)^n + n(Py+Q)(1y+M)(yy+My+N)^{n-1}$$

and after division by $(yy+My+N)^{n-1}$:

$$\begin{aligned} nyy\frac{dM}{dx} + \frac{nydN}{dx} = (2n+1)Pyy &+ (n+1)PMy + PN \\ &+ 2nQy \qquad + nQM. \end{aligned}$$

Hence it must be:

$$\begin{aligned}&\text{I.} && ndM = (2n+1)Pdx\\ &\text{II.} && (n+1)PMdx + 2nQdx\\ &\text{III.} && 0 = PN + nQM.\end{aligned}$$

The first gives

$$P = \frac{ndM}{(2n+1)dx}$$

and the last

$$Q = \frac{-PN}{nM} \quad \text{or} \quad Q = \frac{-NdM}{(2n+1)Mdx},$$

which values substituted in the middle one yield:

$$ndN = \frac{n(N+1)MdM}{2n+1} - \frac{2nNdM}{(2n+1)M}$$

or

$$(2n+1)MdN + 2NdM = (n+1)MMdM,$$

which multiplied by $M^{\frac{-2n+1}{2n+1}}$ and integrated yields:

$$(2n+1)M^{\frac{2}{2n+1}}N = \text{Const.} + (n+1)\int M^{\frac{2n+3}{2n+1}}dM$$

or

$$(2n+1)M^{\frac{2}{2n+1}}N = \text{Const.} + \frac{2n+1}{4}M^{\frac{4n+4}{2n+1}},$$

whence it is

$$N = \alpha M^{\frac{-2}{2n+1}} + \frac{1}{4}M^2.$$

Therefore, because it is

$$Pdx = \frac{ndM}{2n+1} \quad \text{and} \quad Qdx = -\frac{\alpha M^{\frac{-2n-3}{2n+1}}dM}{2n+1} - \frac{MdM}{4(2n+1)},$$

this differential equation:

$$ydy + \frac{nydM}{2n+1} - \frac{MdM}{4(2n+1)} - \frac{\alpha}{2n+1}M^{\frac{-2n-3}{2n+1}}dM = 0$$

is rendered integrable, if it is multiplied by

$$\left(yy + My + \frac{1}{4}M^2 + \alpha M^{\frac{-2}{2n+1}}\right)^n.$$

COROLLARY 1

§97 If it was

$$\frac{-2n-3}{2n+1} = 1 \quad \text{or} \quad n = -1,$$

the differential equation is homogeneous, and if

$$\frac{-2n-3}{2n+1} = 0 \quad \text{or} \quad n = -\frac{3}{2},$$

it is of first order. But in each of both cases there is no difficulty, since the equation can easily be treated.

COROLLARY 2

§98 Therefore, the cases, in which the exponent $\frac{-2n-3}{2n+1}$ is neither 0 nor 1, will be more strange. Therefore, let

$$\frac{-2n-3}{2n+1} = m, \quad \text{whence it is} \quad 2n = \frac{-m-3}{m+1},$$

and the differential equation

$$ydy + \frac{1}{4}(m+3)ydM + \frac{1}{8}(m+1)MdM + \frac{1}{2}\alpha(m+1)M^m dM = 0$$

will be rendered integrable by the multiplicator

$$(yy + My + \frac{1}{4}MM + \alpha M^{m+1})^{\frac{-m-3}{2(m+1)}}.$$

COROLLARY 3

§99 If now for M any functions of x are substituted, one will be able to form such complicated equations, how which have to treated by other methods is hardly clear, although by this method their resolution is obvious.

SCHOLIUM

§100 I anyone wants to continue these investigations, there is no doubt that this method will soon be developed a lot further, so that the whole field of Analysis is significantly promoted. The specimens given here are also of such a nature that they seem to pave the way to more profound investigations, especially if additionally other classes of differential equations are treated in like manner. But these things, I presented up to now, seem to suffice to encourage the Geometers to develop this method further, which goal I had mainly set myself.